Relationships Between Geometric Propositions which Characterise Projective Planes

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Abstract

A projective plane is a point line structure defined by a set of axioms. There exist many geometric propositions which can be used to characterise projective planes, the most well studied of these are Pappus' Theorem from the 4th century and Desargues' Theorem published in 1648. It has been known since the time of David Hilbert that these theorems are not only geometric in nature but are also deeply and fundamentally linked to algebra. For example, Hilbert showed in in his book *Grundlagen der Geometrie* in 1899 that Pappus' Theorem is equivalent to commutative multiplication in the algebraic structure that coordinatises the plane whereas Desargues' Theorem is equivalent to associative multiplication.

The aim of this project is to characterise projective planes by some less well known geometric propositions. Our first example will be Bricard's Theorem, proposed by French engineer Raoul Bricard in 1911. We are motivated specifically by the question 'Does Bricard's Theorem imply Desargues' Theorem?' From a purely geometric perspective this is a fairly elementary gap in our current understanding of projective planes, however it may also reveal shortcuts to information about the algebraic structures which can be used to coordinatise a plane in which Bricard's Theorem holds.

In our attempts to answer this question, we have discovered the crucial role played by the Trilinear Polar Theorem attributed to J.-J.-A. Mathieu in 1865. We have shown that Trilinear Polar is not only a corollary of Bricard's Theorem, but can also be proven as result of Axial Little Pappus in combination with Fano's Axiom. We have also discovered strong evidence to suggest that Bricard's Theorem is actually equivalent to a degenerate version of Desargues' Theorem, suggesting is not equivalent to the full theorem because of a well-known counterexample.

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Chapter 1

Introduction

For thousands of years geometry referred only to the study of Euclidean geometry because it was the only one that was known. In Euclidean geometry, the points and lines must obey a set of five logical and intuitive axioms. They are logical and intuitive because on a small scale Euclidean geometry is an excellent approximation for the physical space we inhabit. All the theorems of Euclidean geometry, many of which became familiar to us in high school, can be directly deduced by arguing using only these five axioms. These days, it is known that there are other geometries of interest, so we need a practical definition that reflects this.

Definition 1.0.1. A **geometry** is a triple $\mathbf{G} = (P, L, I)$, where P is a set whose elements are called **points** and L is a set whose elements are called **lines**. Points are denoted by upper case letters and lines are denoted by lower case letters. Each line can be thought of as a subset of points, then we define an **incidence relation**, I, where $P\ I\ l$ if $P \in l$. In everyday language this is expressed simply as the point P 'lies on' l or the line l 'passes through' P.

We will now introduce a new type of geometry, Projective Geometry, which has its roots in the 15th century. Prior to that little effort had been made to create an illusion of depth or space in Medieval or Gothic art, and scenes were presented in a much more stylized way. That all changed during the Renaissance when artists, driven by a desire to replicate a three dimensional scene on a two dimensional canvas more realistically than ever before, began to use perspective. According to Leonardo da Vinci 'Perspective is nothing else than the seeing of an object behind a sheet of glass, smooth and quite transparent, on the surface of which all the things may be marked that are behind this glass. All things transmit their images to the eye by pyramidal lines' [3].

Imagine, instead of trying to paint a scene you could simply hold up a sheet of glass, trace what you saw and then colour it in. Obviously a skilled painter wouldn't have to trace, they could simply reproduce the image on a canvas but the end result is the same. You would see that instead of appearing parallel as they do on the ground in front of you, lines can be extended to intersect at a focus point on the horizon. It was this revelation that gave us *The School of Athens* by Raphael, *The Last Supper* by da Vinci and *Christ Handing the Keys to St. Peter* by Perugino.

One could also argue that perspective was not only the greatest artistic achievement, but also the greatest mathematical achievement of the Renaissance as it was the genesis of projective geometry. Originally treated as a 'way of doing Euclidean Geometry', it was cemented as a subject in its own right with the publication of 'Traité des propriétés projectives des figures' (Treatise on the projective properties of figures) by Jean-Victor Poncelet (1788–1867) in 1822 [18]. However it had already begun to be formalised in the 17th century by architect and mathematician Girard Desargues (1591-1661) when he proposed one of the most fundamental theorems of projective geometry, Theorem 2.2.1.

In order to introduce the concept of projective geometry we will use Desargues' Theorem as an example. Desargues' Theorem holds in Euclidean space in all cases except where parallel lines are involved, then it requires lengthy modifications. The idea behind projective geometry is instead of modifying the theorem we modify the space, in this case to say that parallel lines do intersect. This can be done fairly simply thanks to David Hilbert's influential work on axiomatisation in his book *Grundlagen der Geometrie* (Foundations of geometry), originally published in 1899 [7]. He defined a geometry as simply the group of theorems that follow from a set of axioms. In this way we can see that projective geometry is just like Euclidean geometry with points, lines and planes etc. except instead of the traditional Euclidean axioms, we will give a new set of axioms.

One important area of projective geometry is the classification of projective planes, as we want to know when two planes are isomorphic, meaning they have the same properties. One tool that we have for classifying projective planes is geometric propositions, which are statements involving points and lines and when they must be incident. Desargues' Theorem, published after his death in 1648 by his friend Abraham Bosse [1], and Pappus' Theorem, which dates back to the 4th century [16], are two such propositions which can be used to classify planes. In this thesis we will be looking at the relationships between these propositions and how their links to algebra can help us understand the relationships further. Variations of these theorems are also of interest and there remain a few open questions which are highlighted along the way as potential areas for future research.

We then introduce another geometric proposition. Named after Raoul Bricard who published it in 1911, very little has yet been discovered about how Bricard's Theorem fits into the wider picture. The aim of this project is to discover how Bricard's Theorem relates to the other geometric propositions, starting with the question 'Does Bricard's Theorem imply Desargues' Theorem?'

Chapters 2 and 3 are an introduction to projective planes and the theorems of Pappus and Desargues. No previous knowledge of projective geometry is required to gain the full benefit of this thesis. Everything you need to know is contained in these chapters.

In Chapter 4 we narrow our focus to projective planes, summarising what is currently understood about their classification. We construct a theorem hierarchy, based on the work of Hessenberg, Pickert and Heyting. We also explore the algebraic significance of each of these theorems and look at the corresponding algebraic hierarchy, focusing on links between the two, discovered by Hilbert, Heyting and Moufang. Lastly, some examples of non-Desarguesian projective planes are introduced.

In Chapter 5 we introduce Bricard's theorem. Bamberg and Penttila (private communication, 2021) give a brief proof of one direction of Bricard's Theorem but do not highlight the fact that their proof implies it follows from Desargues. We expand on this notion and provide a detailed algebraic proof that Desargues' Theorem implies both directions of Bricard. We also show that Bricard fails in a common non-Desarguesian plane, strengthening the relationship between the two theorems. On the other hand, it becomes evident that Bricard's Theorem does not use associative multiplication, suggesting the two theorems may not be equivalent after all.

In Chapter 6 we introduce the Trilinear Polar Theorem proposed by Mathieu in 1865. This chapter contains most of our original results, primarily that Trilinear Polar arises in a new context where it has not been seen before: as a corollary of Bricard's Theorem. We also show that Little Pappus in combination with Fano's Axiom implies Trilinear Polar as well. Significantly we look at whether or not Trilinear Polar implies Bricard as a way of determining whether Bricard's Theorem can possibly imply Desargues. Figure 1.1 gives a summary of our findings where an arrow means 'implies', no arrow means 'does not imply' and a dotted arrow suggests it is still an open problem. Arrows in red indicate original work. It appears that Bricard's Theorem may be split into two directions,$\mathrm{B}^{\mathbf{I}}$ and $\mathrm{B}^{\mathbf{II}}$, which may not necessarily be equivalent. We show in Section 6.2 that D_9, a version of Desargues' Theorem, does imply one specific configuration of Bricard, (1) in Figure 1.1, leading to the conjecture that Bricard's Theorem is equivalent to D_9 and therefore does not imply Desargues' Theorem.

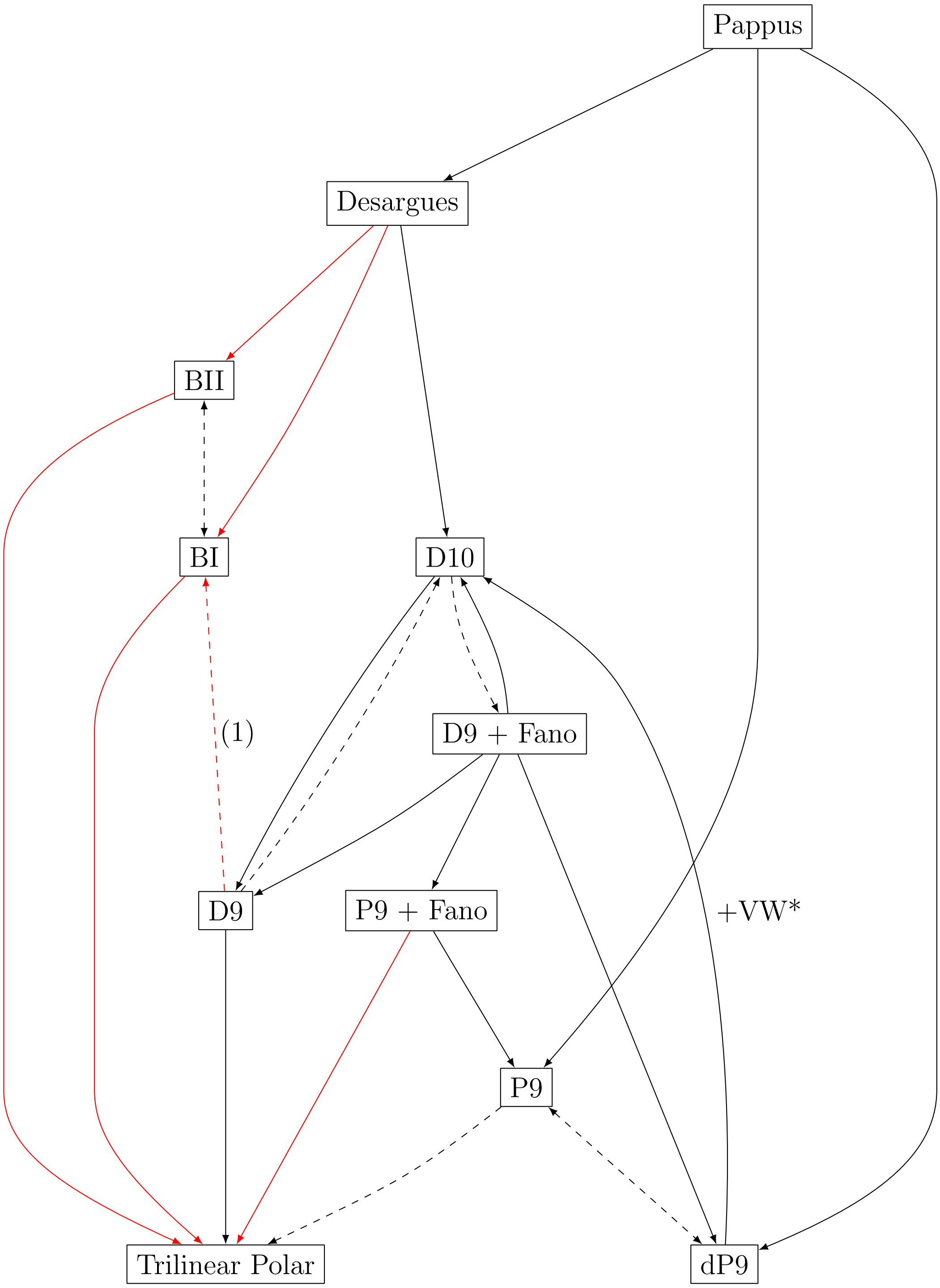


Figure 1.1: Relationships between geometric propositions

Chapter 2

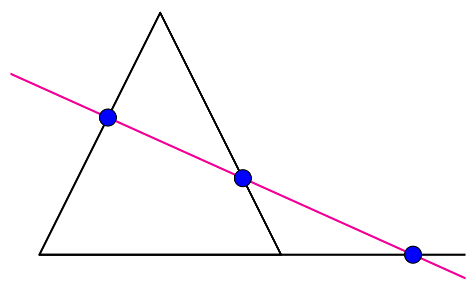
Projective Spaces

2.1 Definition

Definition 2.1.1. A **projective space** is a geometry that satisfies the first three axioms. A projective space is called **nondegenerate** if it also satisfies Axiom 4.

Axiom 1 (line axiom)**.** *For every pair of distinct points there exists a unique line that is incident to both.*

Axiom 2 (Veblen-Young [23])**.** *If a line intersects two sides of a triangle then it also intersects the third.*



Axiom 3. *Every line is incident to at least three points.*

Axiom 4. *There are at least two lines.*

We will assume all the projective spaces discussed from now on are nondegenerate. There are two ways to construct an n-dimensional projective space. The first method is to add a line at infinity, ℓ_∞ to any affine plane, which is a generalization of the well-known Euclidean plane. To explain the concept of a line at infinity start with the idea that pairs of parallel lines meet at a point at infinity, then assume all lines of that parallel class also pass through that point. Next construct a line comprising of all these 'points at infinity' and it is evident this line will intersect every line in the plane, satisfying the axiom that every pair of lines meet in at least one point. This idea can easily be extended to higher dimensions. We may still use the phrase parallel lines throughout, but keep in mind this just means that their intersection lies ℓ_∞.

Definition 2.1.2. A **field** is a set with two operations defined on the set, $(R, +, \cdot)$, usually referred to as addition and multiplication. The operations must obey the following rules:

Commutativity: both addition and multiplication must commute;

Associativity: both addition and multiplication are associative;

Identity: there exists both an additive identity and multiplicative identity;

Inverses: each nonzero element has an additive and multiplicative inverse;

Distributivity: multiplication distributes over addition.

Example 2.1.3. The real, complex and rational numbers with standard addition and multiplication are all examples of infinite fields. There are also fields with finite numbers of elements. Addition and multiplication of each combination of elements can be displayed using a Cayley Table.

The second method of constructing a projective space is to start with an $(n+1)$-dimensional vector space, V, over a field, F. Then $\mathsf{PG}(n, F)$ is the projective space that has the 1, 2, 3 ... dimensional subspaces of V as its points, lines, planes etc. respectively.

Example 2.1.4. Some common examples of infinite projective spaces are real projective space denoted by $\mathsf{PG}(3,\mathbb{R})$, and the projective space over the complex numbers, $\mathsf{PG}(3,\mathbb{C})$.

In general, the second method can also be used to construct a projective space over a noncommutative field.

Definition 2.1.5. A **division ring** satisfies the same axioms field except it does not have commutative multiplication.

Projective spaces over division rings have different geometric properties which allow them to be easily distinguished as we will see in Section 4.3. In some cases, projective spaces over division rings are more interesting, while at other times they make proofs overly complicated and are of no interest. It is always important to be aware of the space in which you are working. In the finite case, there is no difference as any finite division ring is also a field, this will also be explored in Section 4.3.

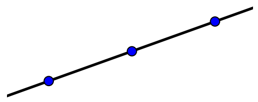
Definition 2.1.6. An **alternative division ring** satisfies the axioms of a division ring without associative multiplication, instead it has a weaker condition on multiplication called **alternativity**:

$$a^{-1}(ab) = b = (ba)a^{-1}$$

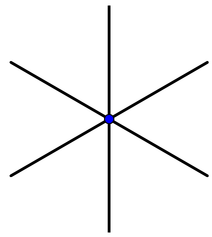
Example 2.1.7. $\mathsf{PG}(3,\mathbb{H})$ is the projective space over the quaternions, a division ring discovered by William Hamilton in 1843. Multiplication of quaternions is noncommutative. An extension of the quaternions known as the octonions was discovered and communicated privately by Hamilton's friend John Graves in 1843, and rediscovered independantly and published by Arthur Cayley in 1845. The octonions are an alternative division ring as multiplication is both noncommutative and nonassociative. Interestingly $\mathsf{PG}(2,\mathbb{O})$ is a projective space but $\mathsf{PG}(3,\mathbb{O})$ or any higher dimension does not exist, the reason for this will become clear in Section 2.2.

We end this introduction to projective spaces with a few definitions. The first two definitions are perhaps the most important of all.

Definition 2.1.8. A set of points are **collinear** if they are all incident to the same line. A set of lines are **concurrent** if they are all incident to the same point.

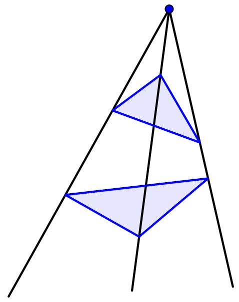


(a) Collinear points

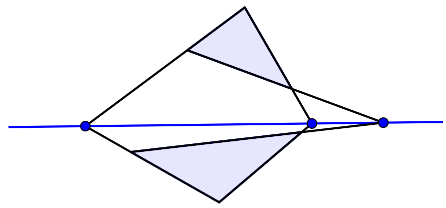


(b) Concurrent lines

Definition 2.1.9. Two triangles are **centrally perspective** if lines through their corresponding vertices are concurrent. Two triangles are **axially perspective** if the intersections of their corresponding sides are collinear.



(a) Centrally perspective



(b) Axially perspective

Definition 2.1.10 (Harmonic conjugate)**.** Given three collinear points A, B, C and a noncollinear point L, let a line through C meet LA at M and LB at N. Let K be the intersection of AN and BM. Then the point $D = LK \cap AB$ is known as the **harmonic conjugate** of C with respect to AB. See Figure 2.3.

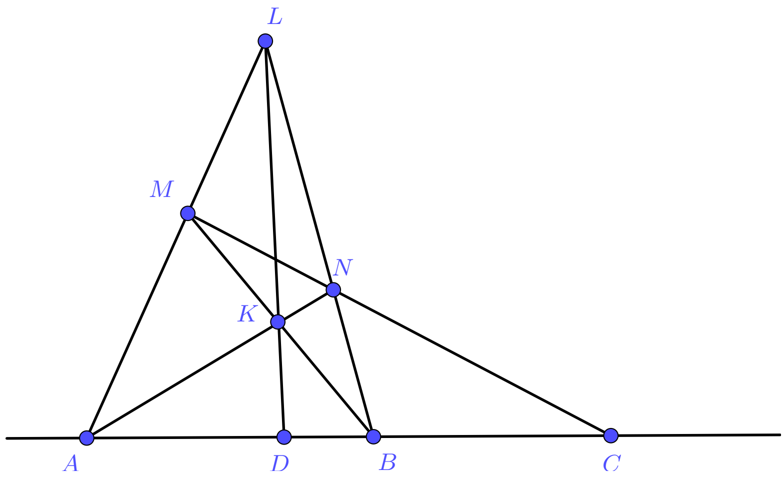


Figure 2.3: Harmonic Conjugate

2.2 Pappus and Desargues

We now introduce two theorems fundamental in the characterisation of projective spaces, the theorems of Pappus (Pappus of Alexandria, ca. 300 A.D.) and Desargues (Girard Desargues (1591-1661)).

Theorem 2.2.1 (Desargues,1648)**.** *Let A, B, C and A', B', C' be two triangles in a projective plane. Then lines AA', BB' and CC' are concurrent if and only if the points $D = AB \cap A'B'$, $E = AC \cap A'C'$ and $F = BC \cap B'C'$ are collinear.*

In other words two triangles are centrally perspective if and only if they are axially perspective.

Example 2.2.2. Desargues' Theorem holds in any projective space of dimension 3 and above. The reasoning for this is explained on page 142 of *The Four Pillars of Geometry* by John Stillwell [22].

Remark. We will see in Section 4.4 that Desargues does not follow from the incidence axioms of a two dimensional projective space. However, in three dimensions the basic incidence properties of points, lines and planes, namely that the intersection of two distinct planes is a line, do imply Desargues' Theorem.

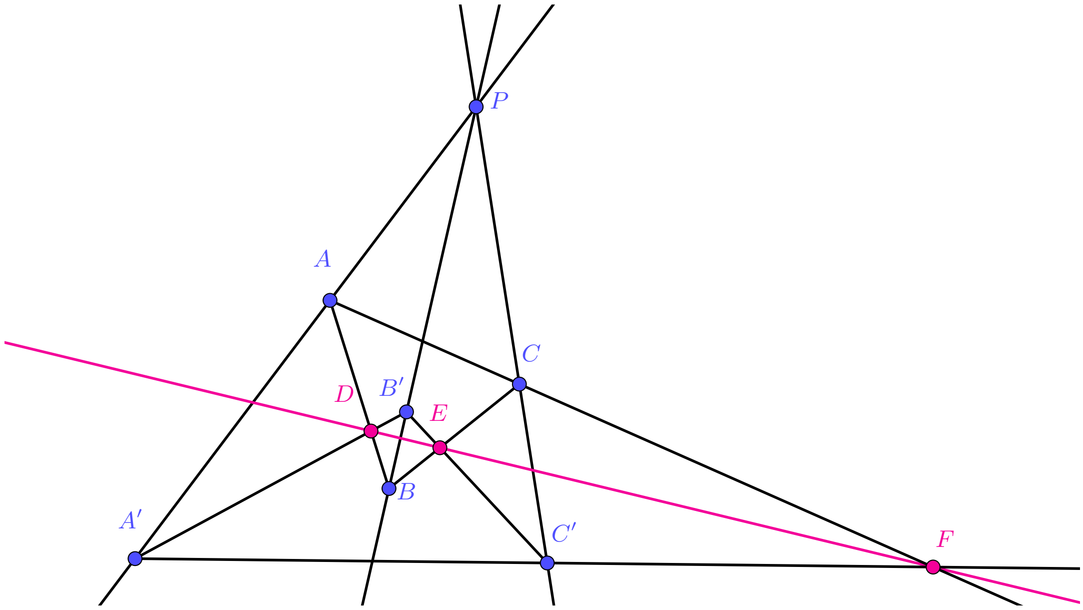


Figure 2.4: Desargues' Theorem

Theorem 2.2.3 (Pappus)**.** *Let g, h be two lines that intersect at point P. If distinct points A, B, C lie on g and distinct points A', B', C' lie on h with $A, B, C, A', B', C' \neq P$ then the points $D = AB' \cap A'B$, $E = AC' \cap A'C$ and $F = BC' \cap B'C$ are collinear.*

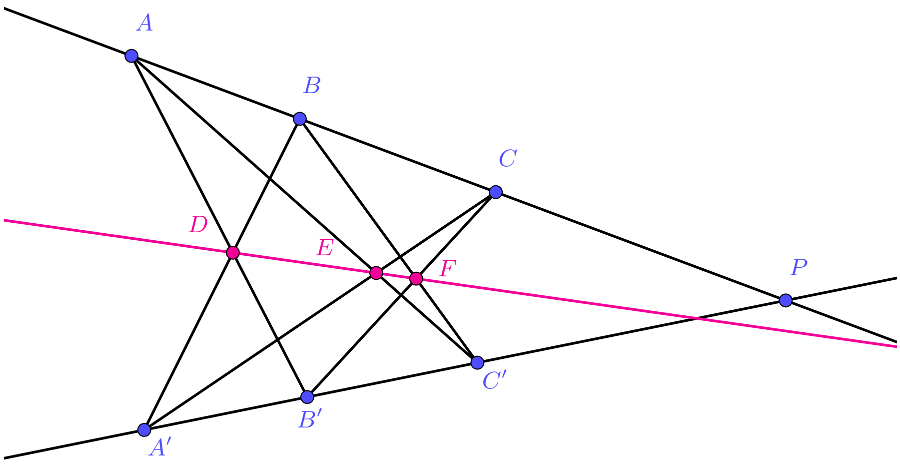


Figure 2.5: Pappus' Theorem

2.3 Homogeneous coordinates

A projective space derived from a vector space can be easily coordinatised, allowing methods from linear algebra to be used. We will demonstrate with $\mathsf{PG}(2,\mathbb{R})$ a two dimensional projective space over the real numbers, but remember the vector space can be over any field or division ring. Since the points of $\mathsf{PG}(2,\mathbb{R})$ are one dimensional subspaces of a three dimensional vector space, they are given by lines through O in $\mathbb{R}^3$, which are uniquely determined up to a scalar by points of $\mathbb{R}^3$. Therefore, a point of the projective space is given by a triple (x, y, z), or any of its nonzero multiples (mx, my, mz). Similarly, a line of $\mathsf{PG}(2, R)$ is given by a plane in $\mathbb{R}^3$, which will satisfy the equation $ax + by + cz = 0$. Multiplying both sides by a scalar $max+mby+mcz = 0$ gives the same plane so lines, like points, are not given by a single triple $[a, b, c]$ but by any of its nonzero multiples $[ma, mb, mc]$. It is clear that a point (x, y, z) lies on a line $[a, b, c]$ if it satisfies the equation $ax+by+cz = 0$. The is the equivalent of the dot product being 0. Also, the intersection of two lines is given by their cross product and the line on two points is given by their cross product. It is important to note that in the case of lines in a projective plane, since it could be over an arbitrary field or division ring, we must assume that multiplication is not necessarily commutative so the coordinates of the line always multiply on the left side of the coordinates of the point. Triples in round brackets denote a point while triples shown in square brackets determine a line.

The four points $(1, 0, 0), (0, 1, 0), (1, 1, 1)$ and $(0, 0, 1)$ are referred to as the Fundamental Quadrangle. We will see later that in projective spaces there are maps which takes groups of points to other groups of points. This means for any proof using homogeneous coordinates, we can make use of the fundamental quadrangle and use a map to show that it holds everywhere, this makes life easier as the fundamental quadrangle produces the simplest arithmetic.

In order to increase familiarity with this process we will give a quick proof of

Pappus' Theorem using homogeneous coordinates.

Proof. We start by making use of the fundamental quadrangle. $A = (1,0,0), B = (0,1,0), A' = (1,1,1)$ and $B' = (0,0,1)$. The line through $AB = [AB|_1, AB|_2, AB|_3]$ goes through $(1,0,0)$ and $(0,1,0)$ therefore it satisfies:

$$AB|_1(1) + AB|_2(0) + AB|_3(0) = 0$$
$$AB|_1(0) + B|_2(1) + B|_3(0) = 0$$
$$\Rightarrow AB|_1 = 0 \text{ and } AB|_2 = 0.$$

So, the line $AB = [0,0,1]$ or $z = 0$. We can then calculate the line $A'B'$ in the same way and find that $A'B' = [1,-1,0]$, or $x = y$. The intersection of these two lines $P = (p_1, p_2, p_3)$ must satisfy:

$$p_3 = 0 \text{ and } p_1 - p_2 = 0$$
$$\Rightarrow p = (p_1, p_1, 0).$$

However up to a scalar this is the same point as $(1,1,0)$ so the intersection of AB with $A'B'$ is $(1,1,0)$. Now C' is any point on $A'B'$ not equal to P so we choose to let $C' = (t,t,1)$, $t \neq 0,1$. It is easy to verify this point does indeed lie on the line $x = y$ but is not the same point as P. Similarly, C is any point on AB not equal to P. Suppose $C = (1,s,0)$, $s \neq 0,1$. We can now calculate the lines $A'B$, AB', $A'C$, AC'. $B'C$, BC' and the points D, E and F.

Firstly we can see from inspection the line AB' is $y = 0$ and $A'B$ is $x = z$. Therefore their intersection $D = (d_1, d_2, d_3)$ can easily be calculated as $(d_1, 0, d_1)$ which is the same point as $(1,0,1)$. Next, the line $BC' = [BC'_1, BC'_2, BC'_3]$ passes through $(0,1,0)$ and $(t,t,1)$.

$$BC'_1(0) + BC'_2(1) + BC'_3(0) = 0 \qquad (B \text{ on } BC')$$
$$BC'_1(t) + BC'_2(t) + BC'_3(1) = 0 \qquad (C' \text{ on } BC')$$
$$\Rightarrow BC' = [1, 0, -t].$$

The line $B'C = [B'C_1, B'C_2, B'C_3]$ passes through $(0,0,1)$ and $(1,s,0)$.

$$B'C_1(0) + B'C_2(0) + B'C_3(1) = 0 \qquad (B' \text{ on } B'C)$$

$$B'C_1(1) + B'C_2(s) + B'C_3(0) = 0 \qquad (\text{ on } B'C)$$

$$\Rightarrow B'C = [-s, 1, 0].$$

Now $F = (f_1, f_2, f_3)$ is the intersection of BC' with $B'C$. So

$$f_1 - tf_3 = 0$$

$$f_1 = tf_3$$

and

$$-sf_1 + f_2 = 0$$

$$f_2 = sf_1.$$

Setting f_3 as 1 gives $f = (t, st, 1)$. The line $A'C = [A'C_1, A'C_2, A'C_3]$ passes through $(1,1,1)$ and $(1,s,0)$.

$$A'C_1(1) + A'C_2(s) + A'C_3(0) = 0 \qquad (C \text{ on } A'C)$$

$$A'C_1 = -A'C_2(s)$$

$$A'C_1(1) + A'C_2(1) + A'C_3(1) = 0 \qquad (A' \text{ on } A'C)$$

$$-A'C_2(s) + A'C_2(1) + A'C_3(1) = 0$$

$$A'C_2(1-s) = -A'C_3$$

$$\Rightarrow A'C = [-s, 1, s-1].$$

Lastly the line $AC' = [AC'_1, AC'_2, AC'_3]$ passes through $(1,0,0$ and $(t,t,1)$:

$$AC'_1(1) + AC'_2(0) + AC'_3(0) = 0 \qquad (A \text{ on } AC')$$

$$AC'_1 = 0$$

$$AC'_1(t) + AC'_2(t) + AC'_3(1) = 0 \qquad (C' \text{ on } AC')$$

$$\Rightarrow AC' = [0, 1, -t].$$

The intersection of these two lines, $E = (e_1, e_2, e_3)$ must satisfy:

$$e_2 - te_3 = 0$$

$$e_2 = te_3$$

and

$$-se_1 + e_2 + (s-1)e_3 = 0$$

$$se_1 = (s+t-1)e_3$$

Setting $e_3 = s$ gives $E = (s+t-1, ts, s)$. In order for these three points to be collinear, then E must be a linear combination of F and D:

$$(s+t-1, ts, s) = (t, st, 1) + (s-1)(1,0,1). \tag{1}$$

This shows that E can be expressed as a linear combination of D and F, therefore D, E and F are collinear. □

Remark. Remember that multiplication is not always commutative so the above proof only holds in the case where $ts = st$ in (1).

Chapter 3

Projective Planes

3.1 Definition

From now on we narrow our focus to the study of projective spaces of geometric dimension 2 only. Since they contain only points and lines, we will call them **projective planes**.

Since a projective plane is a projective space, it satisfies the projective space axioms. However, it turns out that Axiom 2, the Veblen-Young Axiom, can be replaced by a stronger statement which says that every pair of lines meet in exactly one point.

P1. For every pair of distinct points there exists a unique line that is incident to both.

P2. Every pair of lines meet in exactly one point.

P3. Every line is incident to at least three points.

P4. There are at least two lines.

Once again a projective plane can be finite or infinite. In both cases, it can be shown as a consequence of the axioms that there are the same number of points and lines, and every line is incident to the same number of points. The bijection between the points on a line a and another line b will become clear in Section 3.4.

Definition 3.1.1. The **order** of a projective plane is defined as the integer n such that each line is incident to $n+1$ points, each point is incident to $n+1$ lines, and there are n^2+n+1 points and n^2+n+1 lines.

Example 3.1.2. The Fano plane, named after Gino Fano, shown in Figure 3.1, is the smallest example of a finite projective plane. It has order $n = 2$ meaning there are seven points and seven lines. There are three lines through each point and three points on each line.

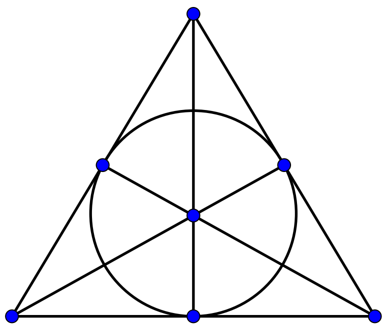


Figure 3.1: Fano Plane

It turns out that all known examples of finite projective planes have prime power order. As an aside, whether or not there exist finite projective planes of nonprime power order is still an open question in finite geometry. The smallest order for which no projective plane has yet been found but existence has not yet been proven impossible is 12. The only other known condition on the order of a finite projective plane is given by the following theorem attributed to Richard Bruck and Herbert Ryser. Interestingly the theorem dates back to 1949 however no further progress has yet been made despite it being an active area of study **??**.

Theorem 3.1.3 (Bruck-Ryser, 1946)**.** *If $n \equiv 1$ or 2 mod 4 and there exists a projective plane of order n, then n must be expressible as the sum of two squares.*

Example 3.1.4. There is no projective plane of order 14 because $14 \equiv 2 \pmod 4$ but it is easy to check that there are no two square numbers that sum to 14.

In addition to the four axioms for a projective plane, call them **P1** - **P4**, we will also give an optional axiom which will turn out to play an important role in the classification of projective planes.

Definition 3.1.5. A **quadrangle** is four points (A, B, C, D), no three of which are collinear. The **diagonal points** of a quadrangle are given by $d_1 = AB \cap CD$, $d_2 = AC \cap BD$ and $d_3 = AD \cap BC$.

Axiom 5 (Fano's Axiom, 1892)**.** *The three diagonal points of a complete quadrangle are never collinear.*

Planes which satisfy Fano's Axiom are called Fano planes. One important yet confusing distinction is that the Fano plane given in Example 3.1 is not a Fano plane in the sense that it does not satisfy Fano's axiom. By taking the bottom two vertices and the midpoints of the two sides as your quadrangle it is easy to satisfy yourself that the diagonal points do lie on a line, the verticle centreline of the triangle. In order to minimise confusion, Fano's Axiom is often referred to as **P6**.

Remark. The reason for **P6** is that some texts (referencekadisonkroman) reserve **P5** to refer to Desargues' Theorem and subsequently **P7** refers to Pappus' Theorem.

Lastly we have another optional condition that is satisfied by some projective planes. It was proposed by Kurt Reidemeister in 1929 [19].

Theorem 3.1.6 (Reidemeister Condition, 1929)**.** *Let A, B and C be three distinct points in a projective plane. Let ℓ be a line through A such that B and C are not on ℓ but $L_1, L_2 \neq A$ are two distinct points on ℓ. Lastly $x, y \neq \ell$ are two more distinct lines through A. Now define $X_1 = x \cap L_1B$, $X_2 = x \cap L_2B$, $Y_1 = y \cap L_1C$ and $Y_2 = y \cap L_2C$. If $Z_1 = CX_1 \cap BY_1$ and $Z_2 = CX_2 \cap BY_2$ then A, Z_1 and Z_2 are collinear.*

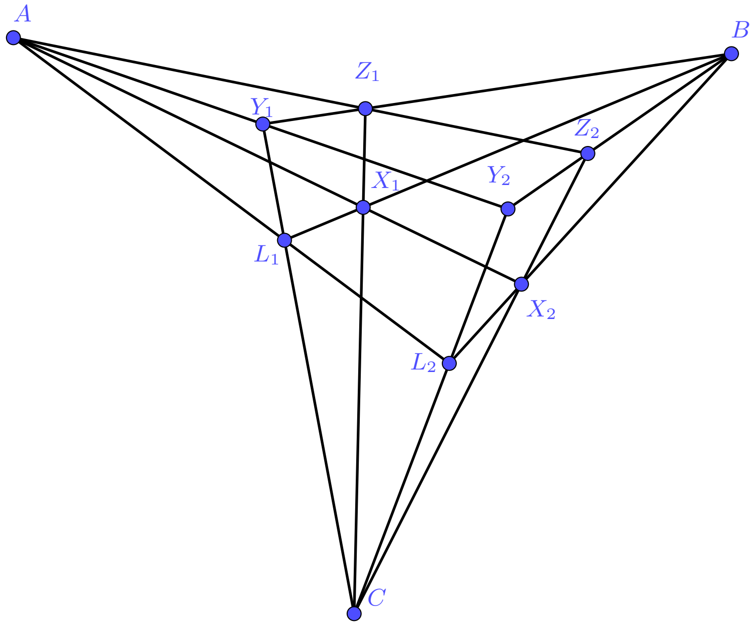


Figure 3.2: Reidemeister Condition

3.2 Duality

One extraordinary property of projective planes is the correspondence between points and lines. Let π be aprojective plane, then taking any theorem which holds in π and interchanging the words 'points' and 'lines', as well as making the necessary grammatical changes produces a dual theorem. Note that the dual theorem will not necessarily hold in π. We practice this idea by recalling **P1**, that every pair of points lies on a unique line. The dual of this statement is that every pair of lines intersect in a unique point which is precisely **P2**. Also, two triangles are centrally perspective if lines through their corresponding vertices are concurrent. The dual of this statement is two triangles are in perspective if the intersection points of their corresponding sides are collinear, which is the definition of axially perspective. The dual of a dual statement is the original statement.

Definition 3.2.1. A theorem is called **self dual** if swapping the terms 'points' and 'lines' produces the same theorem.

Example 3.2.2. Desargues' Theorem says that two triangles are centrally perspective if and only if they are axially perspective. Dualising this theorem we get that two triangles are axially perspective if and only if they are centrally perspective which is exactly the same statement. Therefore Desargues' Theorem is self dual.

Theorem 3.2.3 (Principle of Duality, Chapter 2.4 [21])**.** *If a theorem is deducible from the axioms, then its dual is also deducible from the axioms.*

Simply writing down the dual of each statement used to prove the original theorem will result in a proof of the dual theorem.

Given a projective plane π, consider a bijective map φ which takes the points to lines and lines to points that preserves incidences. For example let P be a point on ℓ then $\varphi(\ell)$ will be a point on the line $\varphi(P)$. This map will create another structure satisfying the axioms of a projective plane. We will call it the **dual** plane, π^*. The dual of any theorem which holds in π will be a theorem in π^*. Also, applying φ twice produces the same mapping as the identity.

Example 3.2.4. Recall that the coordinates of points are given in round brackets and lines are denoted by square brackets. The map taking (a_1, a_2, a_3) to $[a_1, a_2, a_3]$ is called the standard duality.

A plane is called **self dual** if π and π^* are isomorphic. Another way to interpret the Principle of Duality is that the dual of a theorem will hold in π as well as in π^* if and only if it is self dual.

Example 3.2.5. All projective planes $\mathsf{PG}(2, D)$ where D is a division ring are self dual. The reason for this will become clear with Theorem 4.3.1

3.3 Collineations

Definition 3.3.1. A **collineation** is a bijection from a projective plane to itself that preserves lines.

In other words a collineation is a map acting on the points of a projective plane that is an automorphism. For example if we have a collineation φ and points $P_1, P_2, \ldots, P_n$ are collinear then the image of each of the points under φ will still be collinear.

Another important property of projective planes is that the set of all collineations of a projective plane together with an operation which is just the standard composition of maps, forms a group which we denote by $\mathrm{Aut}(\pi)$. Usually, a collineation group refers to a subgroup of $\mathrm{Aut}(\pi)$. Studying the collineation groups of projective planes is another way to classify them.

Definition 3.3.2. Given a group G and a set Ω, a **group action** α is a function $\alpha : G \times \Omega \to \Omega$ such that:

1) $\omega^e = \omega$ where e is the identity of G,

2) $((\omega)^g)^h = \omega^{(g \cdot h)}$, for all g, $h \in G$.

Example 3.3.3. Let $\mathsf{PG}(2, F)$ be the projective plane made from a vector space over F and M be an element of the general linear group, which is the group of 2×2 matrices with entries from F, denoted $\mathsf{GL}(2, F)$. M will induce a collineation on $\mathsf{PG}(2, F)$ and the group action of $\mathsf{GL}(2, F)$ forms a collineation group denoted by $\mathsf{PGL}(2, F)$.

A point is **fixed** by a collineation if $\varphi(P) = P$. Similarly a line is **fixed** if $\varphi(\ell) = \ell$ where ℓ simply refers to the set of points on that line, in any order. In other words the line is fixed set-wise but the points may be permuted around.

Definition 3.3.4. A line fixed point-wise by a collineation is called an **axis**. Here point-wise means that the line is fixed and the points remain in the same order. It is a stronger condition than the line just being fixed. A point through which all the lines are fixed by a collineation is called a **centre**.

A collineation is a **central collineation** if it has a centre.

Theorem 3.3.5 (Piper and Hughes, Theorem 4.9 [9])**.** *A collineation has a centre if and only if it has an axis, and a collineation with more than one centre or axis is the identity.*

Definition 3.3.6. A central collineation is an **elation** if the centre lies on the axis and a **homology** if the centre does not lie on the axis.

Example 3.3.7. Recall that a point is incident with a line if $Ax + By + Cz = 0$ where (x, y, z) are the homogeneous coordinates of the point and $[a, b, c]$ are the coordinates of the line. Consider the following collineation

$$(x, y, z) \mapsto (X + cz, y, z)$$

$$[a, b, c] \mapsto [a, b, c - ac]$$

If a line is incident with $(1, 0, 0)$ it must satisfy $a(1) + b(0) + b(0) = 0$ therefore it has $a = 0$ and has the general form $[0, b, c]$. By the collineation this line is mapped to $[0, b, c - (0)c] = [0, b, c]$ so clearly it is fixed, therefore $(1, 0, 0)$ is the centre. Use the same technique to show every point on the line $[0, 0, 1]$ must be of the form $(x, y, 0)$ which will be mapped to $(x + c(0), y, 0) = (x, y, 0)$ and again it is clear that each of these points is fixed so the line $[0, 0, 1]$ is the axis.

Also $(1, 0, 0)$ is a point of the form $(x, y, 0)$ and so it too lies on the axis, meaning this specific central collineation is an elation.

Definition 3.3.8. Let V be a point and ℓ a line in a projective plane. The plane is (V, ℓ)-**transitive** if for any distinct points A and B not on ℓ with $A \neq B \neq V$ there is a central collineation with centre V and axis ℓ that maps A to B.

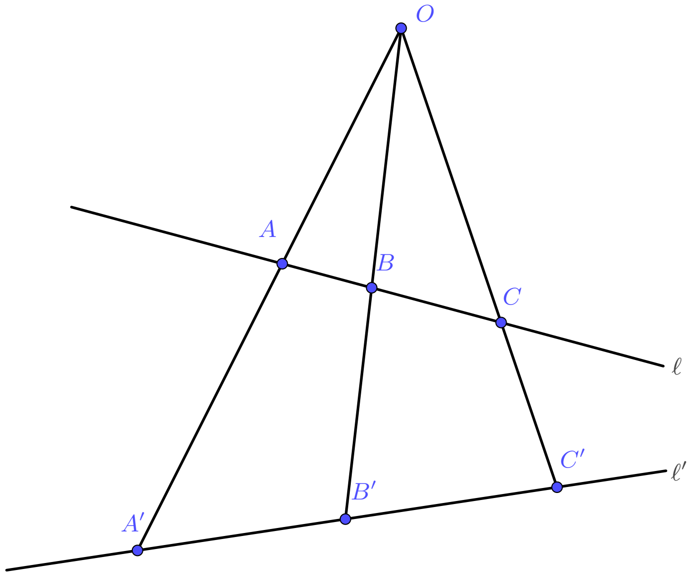


Figure 3.3: Perspectivity with centre O

3.4 Perspectivities and Projectivities

Given a line ℓ and a point O not on ℓ, a **perspectivity** from ℓ to ℓ' is a correspondence of the points on ℓ with the points on ℓ' defined in the following way: if $A \in \ell$ then $A' \in \ell'$ where $A' = AO \cap \ell'$, $Z = \ell \cap \ell'$ gets mapped to itself. The point O is the centre of the perspectivity.

It is a clear consequence of **P1**, that for any two points there is a unique line that is incident to both, that a perspectivity will form a bijection between the points of ℓ and ℓ'. This is why we can be sure there are the same number of points on each line. If points $P_1, P_2, \ldots, P_n$ are sent to $P_1', P_2', \ldots, P_n'$ by a perspectivity with centre O and then to $P_1'', P_2'', \ldots, P_n''$ by another perspectivity with centre O', there may not exist a point which is the centre of a perspectivity directly from $P_1, P_2, \ldots, P_n$ to $P_1'', P_2'', \ldots, P_n''$. Instead, if we wish to transform points $P_1, P_2, \ldots, P_n$ directly to $P_1'', P_2'', \ldots, P_n''$ we can use a projectivity.

Definition 3.4.1. A **projectivity** is a bijective map from the points on a line ℓ to the points on a line ℓ' which can be expressed as the product of a finite number of perspectivities.

It is clear that there exists a projectivity from any two points to any other two points because the intersection of two lines is always a unique point, which can be used as the centre of a perspectivity, but what about any three points? It turns out that this is not always possible. Any three noncollinear points can be projected to any other three noncollinear points and any three collinear points can be projected to any three collinear points.

Theorem 3.4.2 (Fundamental Theorem of Projective Geometry, Kadison and Kromann, Theorem 11.11,[10])**.** *Given three distinct collinear points A, B and C, there is exactly one projectivity which takes them to another three distinct collinear points A', B' and C'.*

3.5 Planar Ternary Rings

We have already seen one method for coordinatising a projective plane over a vector space, those of the form $\mathsf{PG}(2, D)$, where D is a field or division ring. However, not all projective planes are defined in this way. We need another way to coordinatise these other planes, in order to establish whether two given projective planes are isomorphic. It turns out these other planes can be coordinatised using planar ternary rings.

Definition 3.5.1. A **Planar Ternary Ring** or just ternary ring is any set R which includes the elements 0 and 1 together with a ternary operation T which takes any three ordered elements a, b, c of R and prescribes a unique element $T(a, b, c)$ of R such that the following properties hold:

T1. $T(0, b, c) = T(a, 0, c) = c$, for all a, b, $c \in R$.

T2. $T(a, 1, 0) = T(1, a, 0) = a$, for all $a \in R$.

T3. If a, b, c, $d \in R$ with $a \neq c$ then the equation $T(x, a, b) = T(x, c, d)$ has a unique solution for x.

T4. If a, b, $c \in R$ then there is a unique $x \in R$ such that $T(a, b, x) = c$.

T5. If a, b, c, $d \in R$ with $a \neq c$ then the system of equations

$$T(a, x, y) = b,$$

$$T(c, x, y) = d,$$

has a unique solution for (x, y).

Example 3.5.2. A division ring $(R, +, \times)$ together with the ternary operation $T(a, b, c) = ab + c$ satisfies the five conditions and therefore forms a ternary ring.

The ternary ring is denoted by (R, T). Any projective plane of order n can be coordinatised by a planar ternary ring with n elements. A step by step explanation of how this can be done and why it works for a projective planes, not just those defined over a vector space, may be found in *Projective Planes* by Piper and Hughes [9], or *Projective Geometry and Modern Algebra* by Kadison and Kromann [10].

We will now introduce the concept of projective addition and multiplication. Given a ternary operation which takes three elements to one element, define a binary operation called *addition* which takes two elements to one element by the following rule

$$a + b = T(a, 1, b).$$

To interpret this geometrically let $A = (a, a)$ and $B = (b, b)$ be two points. Take the line through A and B and call it the x-axis and take another line and call it the y-axis. Now consider a third line ℓ, for simplicity we will say this line is parallel to the x-axis but recall this just means that their intersection lies on the line at infinity. Then $A + B$ is given by the following construction:

1. Take a line m from A to the intersection of ℓ and the y-axis.

2. Take a line n from B parallel to the y-axis.

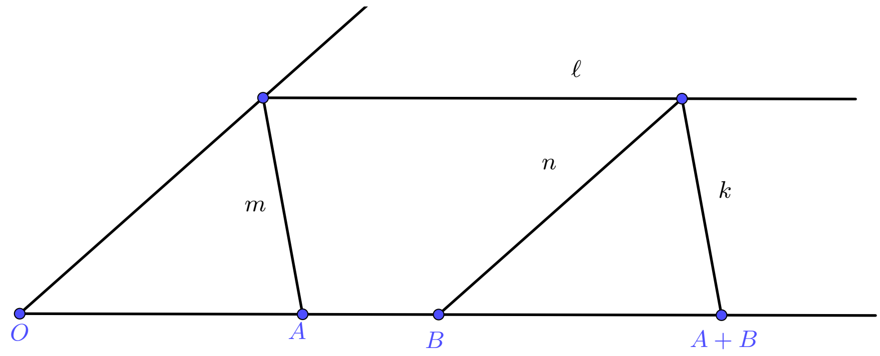


Figure 3.4: Projective addition

3. Take a line k through the intersection of n and ℓ which is parallel to m

Now $A+B$ is the intersection of k with the x-axis.

Multiplication is also a binary operation defined by

$$a \cdot b = T(a, b, 0).$$

Again to interpret this geometrically let $A = (a, a)$ and $B = (b, b)$. Take the line through A and B and call it the x-axis and another line called y-axis and let their intersection be O. Choose a point on the x-axis other than O to denote by 1_x, and choose a different point on the y-axis other than O to denote by 1_y.

1. Take a line m from 1_x to 1_y.

2. Take a line n from A to 1_y.

3. Take a line k from through of B which is parallel to m.

4. Take a line j through the intersection of k and the y-axis which is parallel to n.

Now $A \cdot B$ is the intersection of j with the x-axis.

Definition 3.5.3. A **loop** is a nonempty set G together with a binary operation $\cdot$ such that:

(a) If a, $b \in G$ then $a \cdot x = b$ has a unique solution for x.

(b) If a, $b \in G$ then $y \cdot a = b$ has a unique solution for y.

(c) G has an element e such that $e \cdot x = x \cdot e = x$ for all $x \in G$. The element e is referred to as the identity.

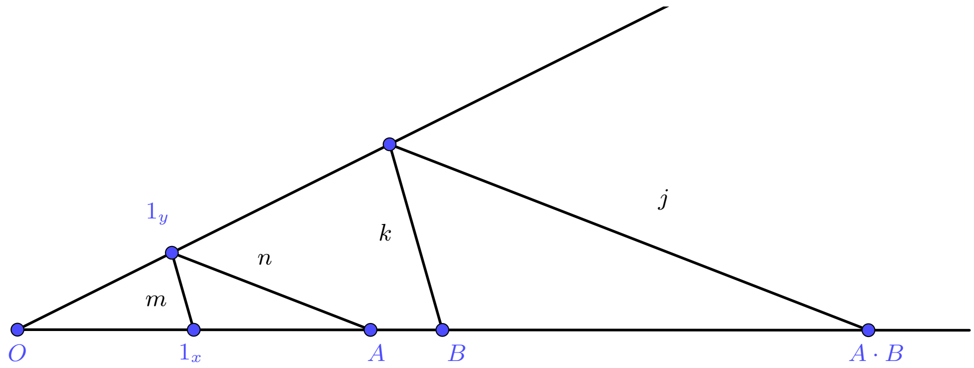


Figure 3.5: Projective multiplication

Example 3.5.4. If (R, T) is a planar ternary ring then $(R, +)$ forms a loop with 0 as its identity and $(R\backslash\{0\}, \cdot)$ forms a loop with 1 as its identity. Note also that a group is a loop such that every element a has an inverse a^{-1} with $aa^{-1} = a^{-1}a = e$.

Definition 3.5.5. A ternary ring R, T is called a **Veblen-Wedderburn system** if:

VW1. $(R, +)$ forms an abelian group,

VW2. $(R\backslash\{0\}, \cdot)$ forms a loop with 1 as the identity,

VW3. $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$,

VW4. R has right distributivity. i.e. $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

Example 3.5.6. The octonions are a finite alternative division ring. Multiplication of the octonions is defined using a Cayley table as it is both nonassociative and noncommutative. However, on inspection of the Cayley table, it is possible to see that the octonions form a Veblen-Wedderburn system.

Chapter 4

Pappus and Desargues in Projective Planes

4.1 Theorem Hierarchy

In a projective plane, the theorems of Pappus and Desargues are simply propositions which may or may not be true in a given plane. (However they are still referred to as theorems as they were originally studied in real projective space where they are always true). A projective plane in which Desargues' Theorem holds is called **Desarguesian**. A projective plane in which Pappus' Theorem holds is called **Pappian**.

As well as Pappus and Desargues there are also a number of other theorems which follow as direct results. For this reason it would make sense for them to be corollaries, however they are still referred to as theorems in their own right because there are non-Desarguesian planes in which these theorems do hold. This will be covered in more detail in Sections 4.3 and 4.4. The first important thing to note is that Desargues' Theorem is self dual. We have already seen why in Example 3.2.2.

Next we consider a special case of Desargues' Theorem known as Little Desargues' Theorem, where the centre of perspectivity lies on the axis of perspectivity.

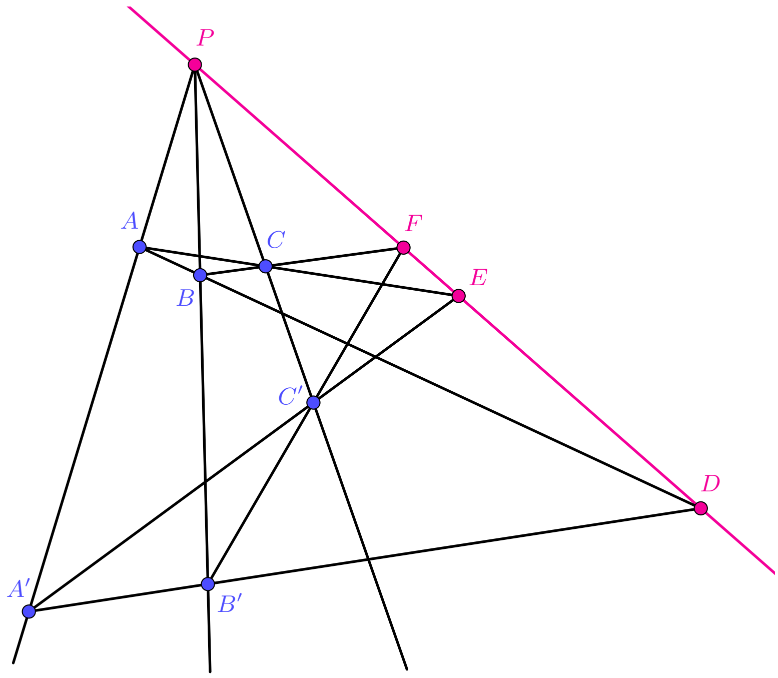


Figure 4.1: Little Desargues

Theorem 4.1.1 (Little Desargues). *If triangles A, B, C and A', B', C' are in perspective from a point P and the points $D = AB \cap A'B'$ and $E = AC \cap A'C'$ are collinear with P, then the point $F = BC \cap B'C'$ is also collinear with P, D and E.*

Another special case arises when one vertex of one triangle lies on the corresponding side of the other.

Theorem 4.1.2 (Special Desargues). *If triangles A, B, C and A', B', C' are in perspective from a point P such that B' lies on AC then $D = AB \cap A'B'$, $E = AC \cap A'C'$ and $F = BC \cap B'C'$ are collinear.*

Desargues' Theorem is referred to as D_{11} because it contains 11 free parameters and Little and Special Desargues' are both referred to as D_{10} because they both involve one extra incidence and therefore have one less free parameter. The nomenclature can be attributed first to Ruth Moufang [14], but was also adopted by Heyting. A projective plane in which D_{10} holds is called a **Moufang** plane.

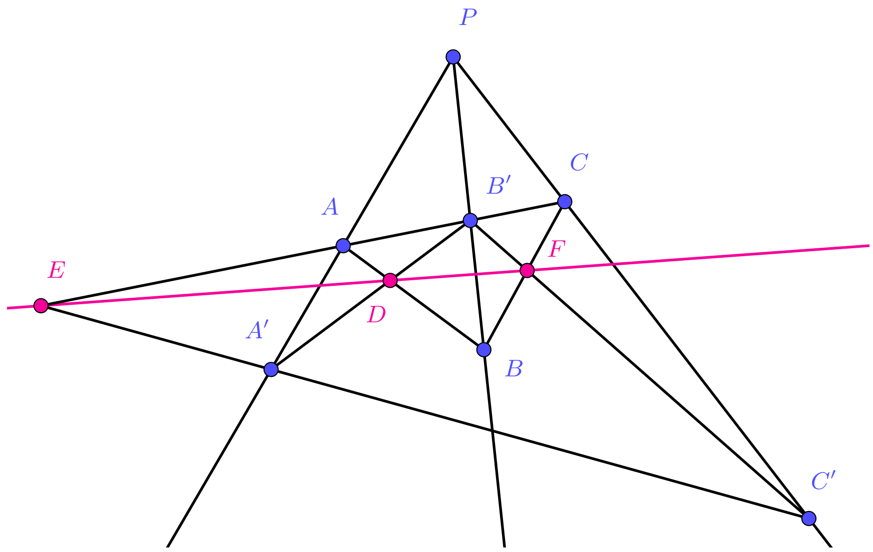


Figure 4.2: Special Desargues

Since the two theorems are referred to by the same term it is important to establish that they are equivalent.

Theorem 4.1.3 (Heyting, Theorem 2.2.8,2.2.10 [6])**.** *Little Desargues and Special Desargues are equivalent.*

Proof. ($\Rightarrow$) Let triangles A, B, C and A', B', C' be in perspective from a point P such that B' lies on AC. We want to show $D = AB \cap A'B'$, $E = AC \cap A'C'$ and $F = BC \cap B'C'$ are collinear. Note triangles DBF and $A'PC'$ are in perspective from B' because D lies on $A'B'$ by the definition of D, B lies on PB' because the original two triangles are in perspective and F lies on $B'C'$ by the definition of F, see Figure 4.2. Also A is the intersection of DB with $A'P$ and C is the intersection of FB with $C'P$ and B' was defined to lie on AC so A, B' and C are collinear. Therefore we have two intersections of corresponding sides being collinear with the centre of perspectivitity so by Little Desargues we know that the intersection of the third pair of corresponding sides is also collinear, call it $G = DF \cap A'C'$. Now we have that G lies on AC by Little Desargues and G also lies on $A'C'$ by definition so G must be the point of intersection of these two lines, but the intersection is unique and was already defined to be E, so $G = E$. We also have that G was on DF by definition therefore E lies on DF, so D, E and F are collinear.

($\Leftarrow$) Let triangles A, B, C and A', B', C' be in perspective from a point P such that the points $D = AB \cap A'B'$ and $E = AC \cap A'C'$ are collinear with P, we want to show $F = BC \cap B'C'$ is also collinear with P, D and E. Note triangles BPC and $DA'E$ are in perspective from A because D lies on AB by definition, A' lies on PA because the two original triangles are in perspective and E lies on AC by definition. Also we have that P lies on ED by construction, see Figure 4.1. Therefore we have two triangles in perspective from a point, one of which has a vertex lying on a side of the other, so by Special Desargues we know the intersections of their corresponding sides are collinear. The intersection of BP with $A'D$ is B', $PC \cap A'E$ is C' and let $BC \cap ED = G$, so $B'C'G$ are collinear. However G lies on BC and $B'C'$ but we already defined their unique intersection to be F so $G = F$. We also know that G lies on ED by definition which means that F lies of ED so P, D, E and F are collinear. □

We now look at adding another incidence to Desargues' Theorem, two extra in total, this will give us D_9. In the same way as D_{10}, Heyting uses D_9 to refer to a group of theorems, five in total, representing the five different ways to configure Desargues' Theorem with two extra incidences. In fact, there are more than five ways to add two incidences but the others all become trivial. We will focus only on what Heyting calls D_9 and D_9^I. Only these two are known to be equivalent so we will refer to them both as D_9, proofs can be found in Heyting's book, Theorems 2.2.17 and 2.2.18 [6]. The other three are known to follow from D_9 but are not known to imply it, these are still open questions.

Theorem 4.1.4 (D_9)**.** *Let triangles A, B, C and A', B', C' be in perspective from a point P. If:*

a) the point B' lies on AC and B lies on $A'C'$,

or

b) the point A' lies on BC and B' lies on AC,

then $D = AB \cap A'B'$, $E = AC \cap A'C'$ and $F = BC \cap B'C'$ are collinear.

Theorem 4.1.5 (Moufang [14])**.** *Desargues' Theorem implies D_{10}, which implies D_9.*

We don't give a formal proof because it is fairly obvious, it follows as an immediate consequence of the fact that D_9 is just one specific configuration of D_{10} and D_{10} is one specific configuration of Desargues. On the other hand, are there any planes in which the reverse of this theorem hold? Is it possible that any of the degenerate versions of Desargues' Theorem imply stronger ones?

Theorem 4.1.6 (Heyting, Theorem 2.4.6 [6])**.** *In a projective plane satisfying* ***P6****, D_9 implies D_{10}.*

Remark. It is not known whether D_9 implies D_{10} in general, this is still an open problem. However, Heyting is able to prove the following Harmonic Proposition: If D is the harmonic conjugate of C with respect to AB and D is never equal to C, then D_{10} is equivalent to D_9. We wish to refer the reader to the proof of Theorem 2.4.6 in Heyting. Therefore it suffices for us to show that this condition on the harmonic conjugate is equivalent to **P6**.

Proposition 4.1.7. ***P6*** *is equivalent to the harmonic conjugate of a point never being equal to itself.*

Proof. Let A, B, C, and D be four points, no three of which are collinear. **P6** says that $E = AD \cap BC$, $F = AC \cap BD$ and $G = AB \cap CD$ are never collinear. This means that $EF \cap CD$, which is precisely the harmonic conjugate of G with respect to CD is never G. Conversely let the harmonic conjugate of G with respect to CD be a point H not equal to G. Then $H = EF \cap CD$. Since two lines intersect in a unique point then EF cannot pass through G as G also lies on CD. □

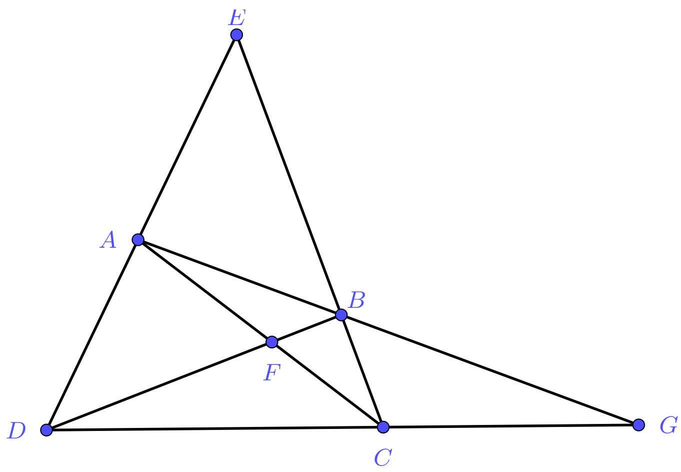


Figure 4.3: **P6** $\iff$ Heyting's Harmonic Proposition

This is the only general case where a weaker theorem is known to imply a stronger one, and it still requires the addition of Fano's Axiom. We can be sure that D_{10} does not imply Desargues' Theorem because there exists a well known counter example which proves it, this will be explored in more detail in Section 4.4. However, it is still possible that D_9 implies D_{10} without Fano's Axiom.

Similarly, a special case of Pappus' Theorem arises when there is an extra incidence, in this case where the line through DEF also passes through P. This is known as Central Little Pappus' Theorem and may be referred to as P_9.

Theorem 4.1.8 (Central Little Pappus)**.** *Let g, h be two lines that intersect at point P. If distinct points A, B, C lie on g and distinct points A', B', C' lie on h with $A, B, C, A', B', C' \neq P$ and the points P, $D = AB' \cap A'B$ and $F = BC' \cap B'C$ are collinear, then the point $E = AC' \cap A'C$ is also collinear with P, D and F.*

The dual of P_9 is referred to as Axial Little Pappus or dP_9.

Theorem 4.1.9 (Axial Little Pappus)**.** *Let g, h be two lines that intersect at point P. If distinct points A, B, C lie on g and distinct points A', B', C' lie on h with $A, B, C, A', B', C' \neq P$ and $E = AC' \cap A'C$ lies on BB', then the points $D = AB' \cap A'B$, E and $F = BC' \cap B'C$ are collinear.*

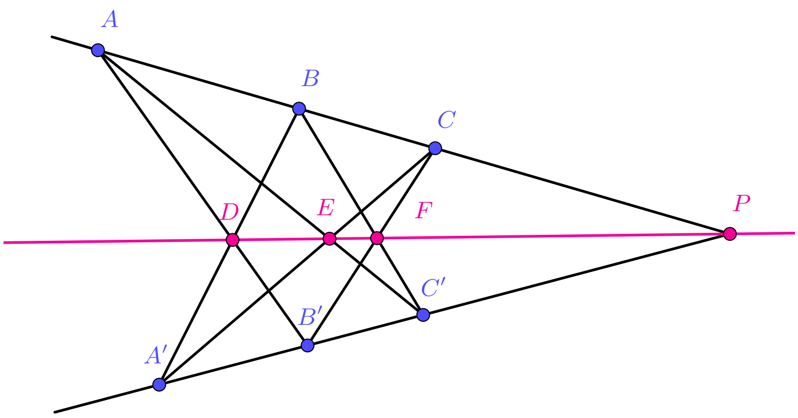


Figure 4.4: Central Little Pappus

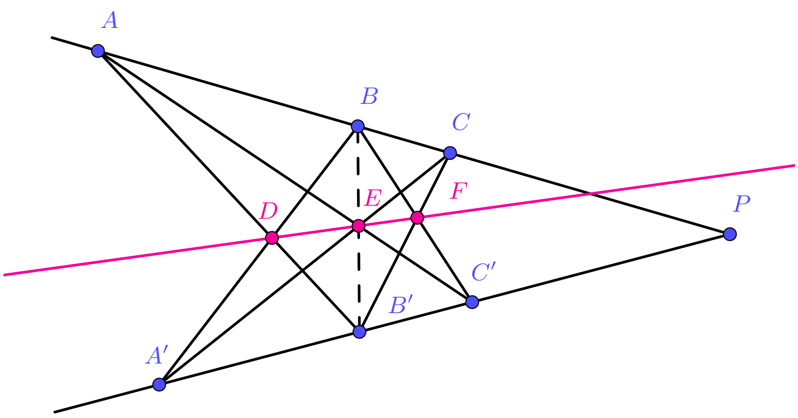


Figure 4.5: Axial Little Pappus

Again, it is easy to see that both of these theorems follow as a direct result of Pappus' Theorem, however it is not known whether they are equivalent to one another. This is another major open question in the area.

4.2 Between Pappus and Desargues

Now that we know the structure of the Desargues chain and the Pappus chain separately, it is natural to ask how the two chains fit together.

Theorem 4.2.1 (Hessenberg [20])**.** *In any projective plane, π, if Pappus' theorem holds then Desargues' Theorem also holds.*

Remark. The proof is not given here but it is geometric in nature. It was originally attributed to Gerhard Hessenberg who published the result in 1905. Although he was correct his original proof didn't cover all possible configurations and was incomplete. The first complete proof was given by Arno Cronheim in 1953.

Theorem 4.2.2 (Heyting, Theorem 2.6.9 [6]). *D_9 implies P_9 and dP_9.*

Proof. We known that the Principle of Duality holds in a plane satisfying D_9 (refHeyting). Therefore, we just need to prove D_9 implies P_9 and then dP_9 will follow by duality. Suppose ABC and $A'B'C'$ are two triangles in perspective from a point P such that B' lies on AC and B lies on $A'C'$. As a result, $AB'C$ and $A'BC'$ are two lines which meet in a point P'. We want to show that if $D = AB \cap A'B'$ and $F = BC \cap B'C'$ are collinear with P' then so is $E = AC' \cap A'C$. Now triangles ABC and $A'B'C'$ are in perspective from a point and each have one vertex lying on a side of the other so by $D9$ the intersections of corresponding sides are collinear, which is precisely the points D, F and P'. If this is the case we can now consider triangles ABC' and $A'B'C$. Once again they are clearly still in perspective from P and since A lies on $B'C$ and A' lies on BC', they each have a vertex lying on a side of the other. This means by D_9 the intersections of their corresponding sides are also collinear. In this case it is the points D, E and P', but F also lies on DP'. We now have that whenever two points out of D, E and F are collinear with P', it can be shown that the third one is also collinear. □

Theorem 4.2.3 (Pickert, Theorem 2.3.20 [17]). *In a finite projective plane Axial Little Pappus implies the Reidemeister Condition.*

Remark. The proof can be found in Pickert Section 2.3 Theorem 20. It follows from algebraic conditions on the loop implied by Axial Little Pappus which Pickert also refers to as the *Thomsen Condition* after Gerhard Thomsen who did some in-

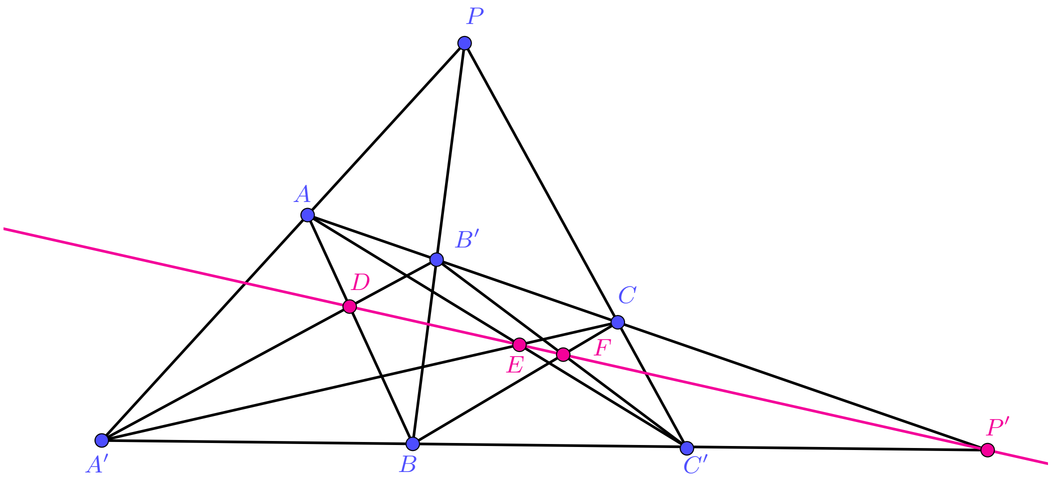


Figure 4.6: D_9 implies Axial Little Pappus

fluential work on the subject in the early 1930s. More on the algebraic consequences of each of these geometric propositions will be covered in Section 4.3.

Theorem 4.2.4 (Kallaher 1967 [11])**.** *In a projective plane where the coordinatising ternary ring is a Veblen-Wedderburn system with the condition*

(2) $$(xy)(zx) = (x(yz))x, \text{ for all } x, y, z \in R,$$

Axial Little Pappus implies D_{10}.

Remark. This result was published in a paper by Michael Kallaher in 1967. His main result was to show that a Reidemeister plane is a Moufang plane if it is coordinatized by a Veblen-Wedderburn system satisfying (2). This result then follows as a corollary thanks to Theorem 4.2.3.

Since D_9 is implied by all the other versions of Desargues' Theorem, that puts P_9 and dP_9 at the very bottom of the picture as the weakest two theorems while Pappus implies everything so it sits right at the top as the strongest. This vertical hierarchy will be reinforced in the following section as it is mirrors the algebraic properties of the structures coordinatising the planes.

Known relationships between geometric propositions are summarised in Figure 4.7. Each arrow in the diagram can be attributed to one of Hessenberg, Pickert and Heyting, where an arrow means 'implies', no arrow means 'does not imply' and a dotted arrow suggests it is still an open problem.

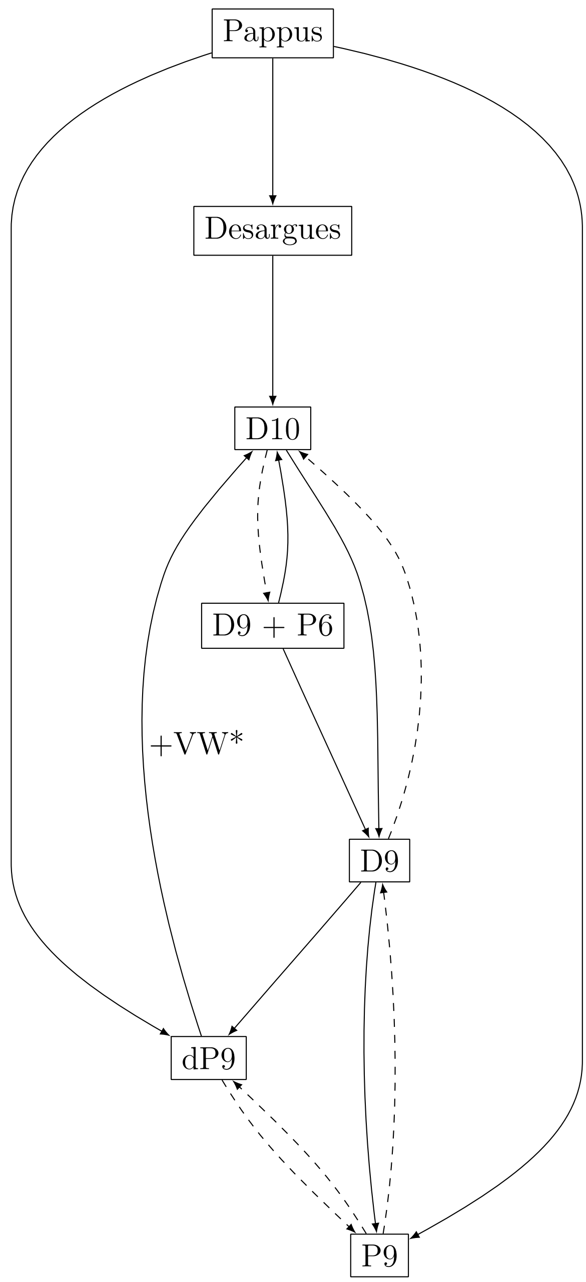


Figure 4.7: Known Relationships between Geometric Propositions

4.3 Algebraic Consequences

It turns out that the veracity of these geometric propositions in a projective plane reveals important information about the algebraic structure of the planar ternary rings that coordinatise it, and so they can be used to characterise projective planes. Since we know the geometric theorem hierarchy we can also construct a corresponding algebraic structure hierarchy. The first major result is traditionally attributed to Hilbert.

Theorem 4.3.1 (Hilbert [7]). *A projective plane is isomorphic to* $\mathsf{PG}(2, D)$*,* D *a division ring, if and only if it satisfies Desargues' theorem.*

Proof. ($\Leftarrow$) A plane isomorphic to $\mathsf{PG}(2, D)$ can be coordinatised using homogeneous coordinates. Therefore we can prove Desargues' Theorem algebraically, like we did with Pappus' Theorem earlier. Again we start by making use of the Fundamental Quadrangle. Let $B = (1, 0, 0)$, $C = (0, 1, 0)$, $B' = (1, 1, 1)$ and $C' = (0, 0, 1)$. Now $P = BB' \cap CC'$. We can see by inspection that BB' is the line $y = z$ and CC' is the line $x = 0$. Therefore, their intersection is of the form $(0, s, s)$ for some $s \in D$, which is the same point as $(0, 1, 1)$. A is a point not on BB' or CC', so we are free to choose $A = (1, 0, 1)$. Now the line PA has:

$$PA_1(0) = PA_2(1) + PA_3(1) = 0 \qquad (P \text{ on } AP)$$

$$PA_1(1) = PA_2(0) + PA_3(1) = 0 \qquad (A \text{ on } AP)$$

$$\Rightarrow PA_1 = PA_2 = -PA_3$$

So the line $PA = [1, 1, -1]$. This is also the line $x + y = z$. Triangles ABC and $A'B'C'$ will be in perspective from P if A' is any point on PA so let $A' = (1, m, 1+m)$. Now we are ready to calculate $D = AB \cap A'B'$, $E = AC \cap A'C'$ and $F = BC \cap B'C'$. Firstly by inspection we see that BC is the line $z = 0$ and $B'C'$ is the line $y = x$ so $D = (1, 1, 0)$. Next, we can easily see that AC is the line $x = z$ while the line $A'C'$

has:

$$A'C_1'(0) + A'C_2'(0) + A'C_3'(1) = 0 \qquad (C' \text{ on } A'C')$$

$$A'C_1'(1) + A'C_2'(m) + A'C_3'(1+m) = 0 \qquad (A' \text{ on } A'C')$$

$$A'C_1' = -A'C_2'(m)$$

So $A'C' = [m, -1, 0]$ or $mx = y$. The intersection of $x = z$ and $mx = y$ is the point $(1, m, 1)$, so $E = (1, m, 1)$. Lastly AB is the line $y = 0$ and $A'B'$ has:

(1) $$A'B_1'(1) + A'B_2'(1) + A'B_3'(1) = 0 \qquad (B' \text{ on } A'B')$$

(2) $$A'B_1'(1) + A'B_2'(m) + A'B_3'(1+m) = 0 (A' \text{ on } A'B')$$

$$A'B_2'(m-1) + A'B_3'(m) = 0 \qquad ((2) \text{ subtract } (1))$$

$$A'B_2'(m-1) = -A'B_3'(m)$$

and

$$A'B_1' = -A'B_2' - A'B_3'$$

From this we get that $A'B' = [-1, m, 1-m]$. The intersection of AB with $A'B'$ is therefore $(1-m, 0, 1)$ so $F = (1-m, 0, 1)$. In order for D, E and F to be collinear, we must be able to express E as a linear combination of the coordinates for D and F. Note that

$$(1, m, 1) = m(1, 1, 0) + (1-m, 0, 1).$$

This implies that D, E and F are collinear.

($\Rightarrow$) Recall the description of projective addition and multiplication defined in Section 3.5. Once the concepts of projective addition and multiplications are defined we can prove, for example, that multiplication is associative using Desargues' Theorem.

Regard Figure 4.10. Lines of the same colour are parallel, remembering that this just means they intersect at a point on ℓ_∞. Consider the triangle with the vertices P, (AB) and $(AB)1_y \cap BP$ and the triangle with vertices Q, $A(BC)$ and $Q(BC) \cap R(A(BC))$. Each of these triangles has a pink side, a blue side and a black

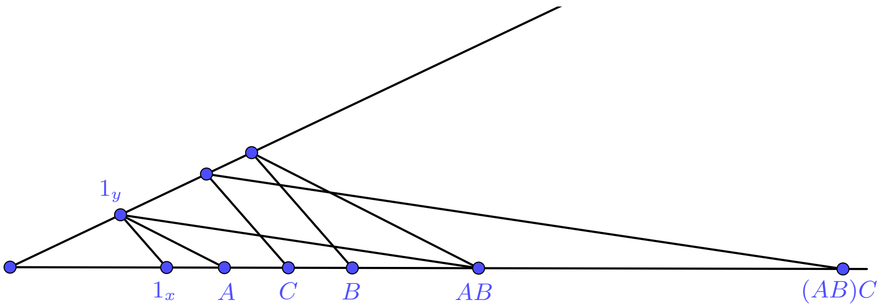


Figure 4.8: Construction of $(AB)C$

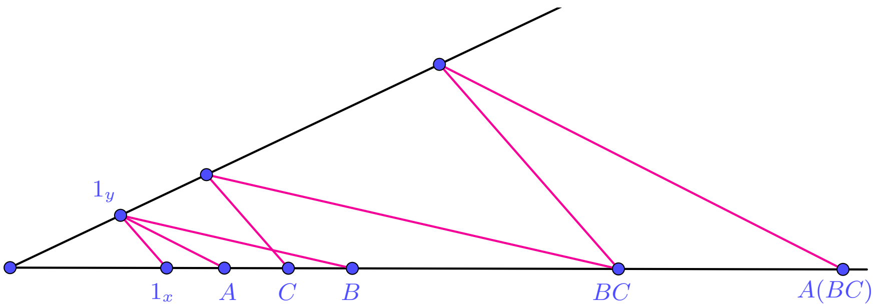


Figure 4.9: Construction of $A(BC)$

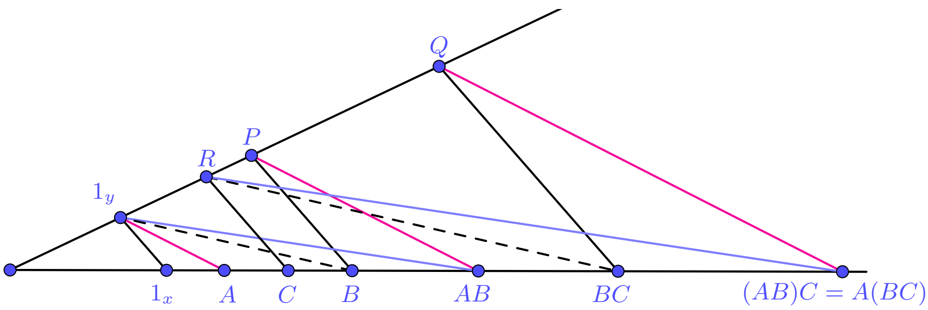


Figure 4.10: $(AB)C = A(BC)$ because of Desargues' Theorem

side. Since the intersections of their corresponding sides lie on a line, ℓ_∞, by the converse of Desargues' Theorem, the two triangles must be in perspective from a point which implies the intersection of the pink and blue lines in the second triangle intersect on the x-axis. Since the intersection of the blue line with the x-axis is $(AB)C$, from Figure 4.8, and the intersection of the pink line with the x-axis is $A(BC)$, this implies that $(AB)C$ and $A(BC)$ are the same point.

Desargues' Theorem can be used to prove all but one of the nine field axioms using techniques similar to the one above. The only law it fails to prove is commutativity, in fact we need Pappus' theorem for that, but a field in which multiplication is noncommutative is a division ring. Therefore, if Desargues' Theorem holds in a projective plane, it can be shown that it is isomorphic to a vector field over a division ring. □

Example 4.3.2. The theorem of Desargues holds in any projective plane defined over a division ring, any projective space of dimension other than 2 and in any Pappian projective plane. It also holds any finite plane with characteristic.

Theorem 4.3.3 (Moufang 1948 [14])**.** *A projective plane is isomorphic to* $\mathsf{PG}(2, D^*)$*,* D^* *an alternative division ring, if and only if it satisfies Little Desargues' theorem.*

Remark. The proof is not given here but is essentially the same as above. The idea is to use coordinates and argue algebraically to show that if we have alternativity, Little Desargues will hold. The second part is construct both $a^{-1}(ab)$ and $(ba)a^{-1}$ using projective multiplication, and then employ the Little Desargues configuration to show that both of these point coincide with the point B, meaning they are all equal. A full proof can be found in the paper Projectve Planes by Marshall Hall from 1943 [4]. In fact, the theorem is actually attributed to Ruth Moufang who's 1933 paper claimed to prove that D_9 was equivalent to coordinatization by an alternative division ring. However, her proof is based upon an incorrect assumption and she failed to realise that she was actually using Little Desargues' along the way and so she actually proved that $D_{10} \Rightarrow$ alternative division ring.

In a follow up paper she showed that 'alternative division ring' $\Rightarrow D_9$. From there it requires only a simple modification to show 'alternative division ring' $\Rightarrow D_{10}$, which is why Moufang is credited with the full discovery. The mistake was not pointed out until Hall's paper, in which he correctly substituted Little Desargues in place of D_9.

Moufang also repeats her mistake in another paper from 1948, again attempting to show D_9 is equivalent to coordinatisation by an alternative division ring. It is still not known whether this is true. On the other hand, she successfully provides a counterexample to show that the full Desargues' Theorem does not hold in an alternative division ring. Moufang was the first to propose the Octonions as a natural home for Little Desargues but not Desargues. At the time, the Octonions were the only known alternative division ring over the real numbers. It has now been shown that there are no others. This is the reason for which a projective space of dimension 3 and above over the octonions cannot exist. Suppose that it did, then recalling Example 2.2.2, Desargues' Theorem would hold and therefore the plane could be shown to have associative multiplication, a clear contradiction as the octonions are nonassociative.

Theorem 4.3.4 (Heyting, Theorem 3.3.3 [6])**.** *D_9 implies that if $ab = 1$ then $ba = 1$ and for all x $(xa)b = b(ax) = x$ in all ternary rings that coordinatise it.*

Remark. The proof uses an application of Axial Little Pappus, which is fine because P_9 follows from D_9, and then uses harmonic pairs in projective multiplication to construct $(xa)b$ and show that it is equal to $b(ax) = x$.

Theorem 4.3.5 (Hilbert [7])**.** *A projective plane is isomorphic to* $\mathsf{PG}(2, F)$*, F a field, if and only if it satisfies Pappus' theorem.*

Proof. By Theorem 4.2.1, Pappus' Theorem implies Desargues' Theorem. Since we know Desargues' Theorem implies all the field axioms except commutativity, the second stage is to show that in a projective plane π, Pappus' theorem is the equivalent of commutative multiplication. This can clearly be seen from the proof of Pappus given in Section 2.3. If Pappus' Theorem holds then st must be equal to ts, meaning multiplication is commutative so π is isomorphic to $\mathsf{PG}(2, F)$. If π is isomorphic to $\mathsf{PG}(2, F)$ then multiplication must be commutative, therefore $st = ts$, in which case the points D, E and F are collinear, which proves that Pappus' Theorem must hold. □

Example 4.3.6. The theorem of Pappus holds in any projective plane defined over a field but fails in a projective plane defined over an structure that does not have commutative multiplication, for instance a division ring.

Theorem 4.3.7 (Pickert, Theorem 5.4.18 [17]). *In a projective plane which satisfies Central Little Pappus, every ternary ring has an additive inverse for each element and a multiplicative inverse for each nonzero element.*

Remark. The proof can be found in Pickert Section 5.4 Theorem 18. He also adds a third equivalent statement which is simply that in every ternary ring each nonzero element has a multiplicative inverse.

Theorem 4.3.8 (Reidemeister, 1929 [19]). *If a projective plane satisfies the Reidemeister Condition, then every ternary ring coordinatising it has associative addition.*

Remark. The proof is in Reidemeister's original paper from 1929 in which he first introduces the condition which is now named after him. Interestingly, the paper is about topological questions in differential geometry yet his results are significant in projective geometry.

Corollary 4.3.9 (Pickert, Theorem 5.4.16 [17]). *In a projective plane which satisfies Axial Little Pappus every ternary ring which coordinatises it has associative and commutative addition.*

Remark. For the proof see Pickert Section 5.4 Theorem 16.

Theorem 4.3.10 (Kadinson and Kromann, Theorem 8.23 [10]). *If P6 holds in a Pappian projective plane then it is isomorphic to* $\mathsf{PG}(2, F)$ *where* F *is a field of characteristic* $\neq 2$.

Proof. Again we use a collineation to take any four points to the fundamental quadrangle $A = (1, 0, 0), B = (0, 1, 0), C = (1, 1, 1)$ and $D = (0, 0, 1)$. By inspection we can see that

AB: $z = 0$	CD: $x = y$	BC: $x = 0$
AD: $y = z$	AC: $y = 0$	BD: $x = z$

From here we can see the the intersection of $AB \cap CD$ is $(1, 1, 0)$, $AC \cap BD$ is $(1, 0, 1)$ and $AD \cap BC$ is $(0, 1, 1)$. Next we calculate the line through $(1, 1, 0)$ and $(1, 0, 1)$.

$$A(1) + B(1) + C(0) = 0$$

and

$$A(1) + B(0) + C(1) = 0$$

$$\Rightarrow A = -B = -C$$

This gives us that the line is $[1, -1, -1]$. Now, if the third diagonal point, $(0, 1, 1)$ is on this line then we have

$$1(0) - 1(1) - 1(1) = 0$$

$$\Rightarrow 2 = 0,$$

which implies the characteristic of the field is 2, so **P6** holds only when the characteristic of the field is $\neq 2$.

One the other hand suppose the charactersitic of the field is 2, the by the calculation above, the three diagonal points are collinear, meaning **P6** does not hold. □

Example 4.3.11. **P6** holds in a projective plane of 13 points, which is the projective plane over a field of 3 elements because it has characteristic $3 \neq 2$. However, as we saw earlier **P6** does not hold in the Fano Plane because it has 7 points and therefore is over a field of 2 elements and has characteristic 2.

The algebraic consequences of geometric propositions are outlined in Figure 4.11. In this form, it is easy to see the clear correlation between the geometric and algebraic hierarchies, for example every Pappian plane is Desarguesian because every field is a division ring. Note however that D_{10}, D_9, P_9 etc. include more conditions on points and lines which must be satisfied, suggesting they are stronger theorems but in fact they imply weaker algebraic conditions.

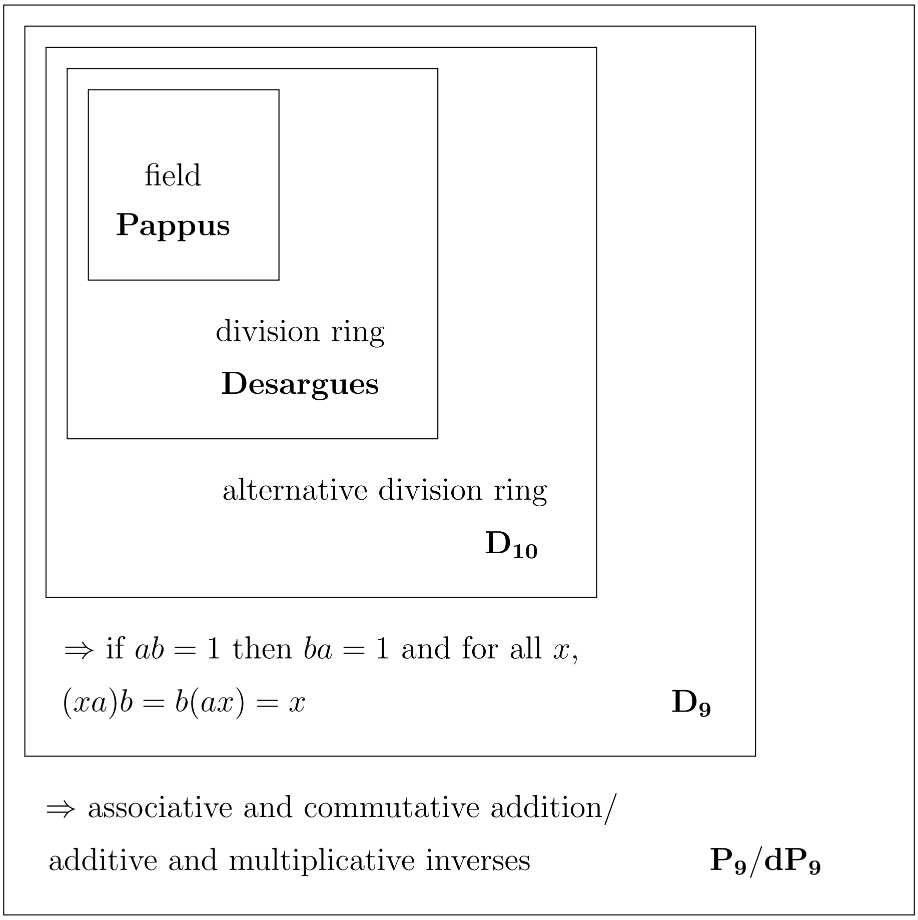


Figure 4.11: Corresponding algebraic conditions

Of course, in the finite case things look quite different. In fact, they are much simpler thanks to a few results given below. Firstly we will use Wedderburn's Theorem, sometimes also referred to as Little Wedderburn's Theorem, from 1905. It is a very famous algebraic result for which many different proofs can easily be found. Secondly we use the Artin-Zorn Theorem from 1930 which is a generalization of Wedderburn's Theorem.

Theorem 4.3.12 (Wedderburn's Theorem 1905). *Every finite division ring is commutative and therefore a field.*

Corollary 4.3.13. *Any finite projective plane which is Desarguesian is also Pappian.*

Proof. By Theorem 4.3.1, any projective plane which is Desarguesian is isomorphic to $\mathsf{PG}(2, D)$. However in the finite case D will also be a field so by Theorem 4.3.5 any plane which is isomorphic to $\mathsf{PG}(2, D)$ is also Pappian, meaning they are equivalent. □

Theorem 4.3.14 (Artin-Zorn Theorem 1930). *Any finite alternative division ring is a field.*

Corollary 4.3.15. *A finite Moufang Plane is Desarguesian.*

Proof. A Moufang plane is equivalent to the vector space being over an alternative division ring. By the Artin-Zorn theorem a finite Moufang plane is equivalent to a vector space over a field which by Theorem 4.3.5 is Pappian and therefore also Desarguesian. □

Lemma 4.3.16 (Reidemeister [19]). *In a finite projective plane P_9 and dP_9 are equivalent.*

Theorem 4.3.17 (Lüneburg 1960 [12]). *A finite projective plane satisfying either version of Little Pappus' Theorem is Pappian.*

Remark. This result is due to Heinz Lüneburg in 1960. Lüneburg proved that a finite projective plane is Desrguesian if it satisfies: **a)** the Little Reidemeister Condition, which is weaker than the full Reidemeister Condition and only implies the associative properties of the additive loop of the planar ternary rings which coordinatise the plane, instead of the ternary rings themselves, and **b)** Axial Little Pappus' Theorem. However by Theorem 4.2.3, in a finite projective plane the Reidemeister Condition and subsequently the Little Reidemeister Condition follow from Axial Little Pappus, meaning condition **a)** above follows from condition **b)**. Since Central and Axial Little Pappus are equivalent in a finite projective plane, it follows from Lüneburg's Theorem that a finite projective plane is Desarguesian if it satisfies either version of Little Pappus' Theorem. Since a finite Desarguesian plane is also Pappian by Corollary 4.3.13 we have that in the finite case Little Pappus implies Pappus.

Therefore we have a collapsing of our theorem hierarchy as they become equivalent to one another which is shown in Figure 4.12.

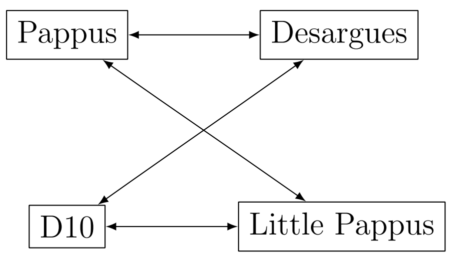


Figure 4.12: Relationships between geometric propositions in finite projective planes

So far we have discussed the algebraic consequences of the geometric properties on the planar ternary rings that can be used to coordinatise them. However, projective planes can also be classified by the algebraic structure of their collineation groups.

Example 4.3.18. A plane is Desarguesian if it is (V, ℓ)-transitive and a Moufang Plane if every line is a translation line.

This method of classification is not our focus, but a full treatment can be found in Piper and Hughes [9].

4.4 Non-Desarguesian Projective Planes

We will now give some examples of both finite and infinite non-Desarguesian projective planes.

Example 4.4.1. The first finite non-Desarguesian plane was discovered by Oswald Veblen and Joseph Wedderburn in 1907, it has order 9.

Theorem 4.4.2 (Hall 1956 [5])**.** *All non-Desarguesian projective planes have order at least 9.*

Remark. No proof is given, instead we refer the reader to the work of Marshall Hall. In 1943, Hall generalised Veblen and Wedderburn's plane of order 9 into an infinite family of non-Desarguesian projective planes. Then, in 1956 he proved the uniqueness of the projective plane of order 8 [5]. Since projective planes of orders less than 8 had already been shown to be unique (apart from the projective plane of order 6 which does not exist) and Desarguesian, Hall's paper also proves that 9 is the smallest possible order of a non-Desarguesian projective plane.

There are also infinite non-Desarguesian projective planes. Perhaps the first class to come to mind is the infinite Moufang planes since we know that D_{10} does not imply Desargues, thanks to the existence of a counterexample.

Example 4.4.3. The projective plane over the octonions is an infinite projective plane which is non-Desarguesian.

More generally infinite alternative rings are not division rings and since Desargues can be shown to be equivalent to associative multiplication, it is clear that Desargues will not hold in an infinite Moufang plane.

Our last example dates back to the work of David Hilbert. In his *Grundlagen der Geometrie*, Hilbert proved that axioms **P1-P4** are a consequence of Desargues' Theorem however the converse is not true [7]. To show that Desargues' Theorem is

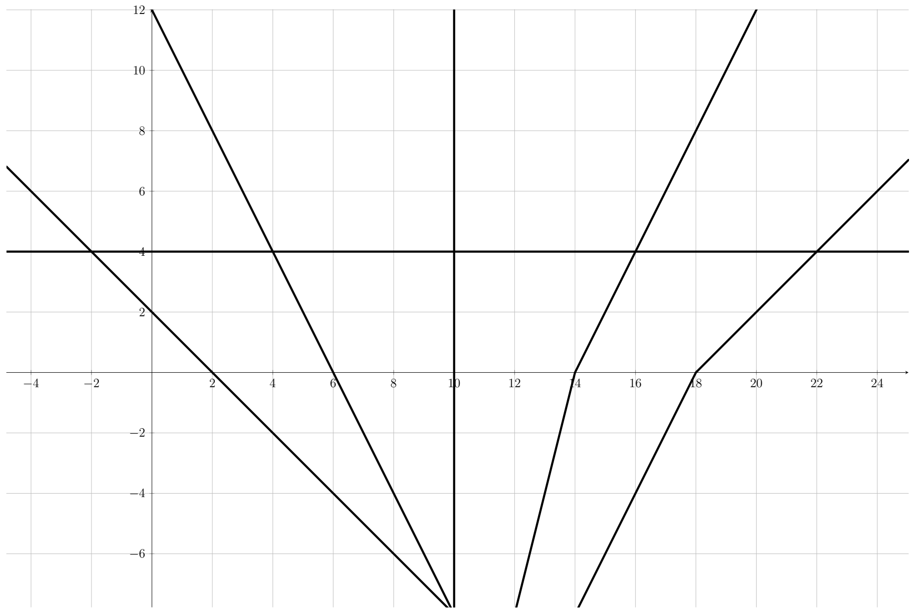


Figure 4.13: The Moulton Plane

not a consequence of the plane axioms, he set out to construct a non-Desarguesian projective plane. However, his synthetic counterexample is far more complicated than necessary. In 1902, Forest Ray Moulton, with the same objective, proposed a much simpler construction of an infinite plane which does not satisfy Desargues' Theorem [15]. It is now called the *Moulton Plane* after him.

Example 4.4.4. The Moulton Plane is made by taking the real affine plane and adding a line at infinity to make it a projective plane. The points are simply the points of $\mathbb{R}^2$. The horizontal, vertical lines as well as the lines with negative gradient remain the same while the lines with positive gradient are refracted by a factor of 2 at the x-axis. A picture of some lines in the Moulton Plane is given in Figure 4.13. It is not immediately obvious that the Moulton Plane is in fact still a projective plane. We will prove it satisfies **P1**, **P2** is shown in a similar manner and **P3** and **P4** are trivial.

Claim 4.4.5. *The Moulton Plane satisfies* ***P1****.*

Proof. To show that for every pair of distinct points there exists a unique line that is

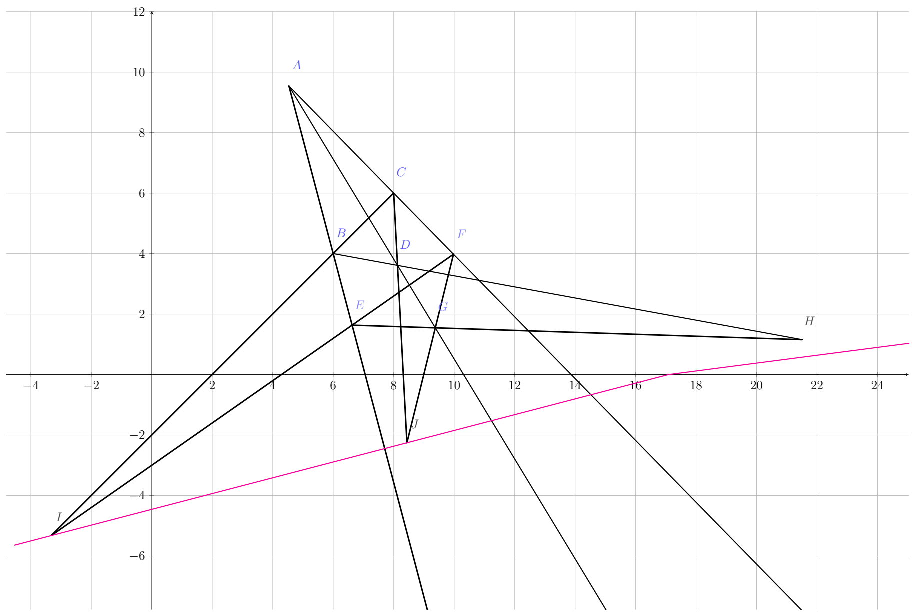


Figure 4.14: Desargues' Theorem fails in the Moulton Plane

incident to both, the only case we need to consider is when the two points straddle the x-axis, and the one above is to the right of the one below, otherwise **P1** is satisfied by the fact that $\mathbb{PR}^2$ is a projective plane. Let A be a point below the x-axis and let B a point above the x-axis to the right of A. Let ℓ be a line through A and B with gradient l_1 below the x-axis and l_2 above. Suppose m is a line through A with gradient m_1 below the x-axis and m_2 above. If $m_1 > l_1$ then that value of the x-intercept of m will be less than that of ℓ and $m_2 > l_2$ meaning m and ℓ cannot intersect above the x-axis. Hence, if B lies on ℓ, m cannot pass through B, instead m will pass to the left of B. Similar logic can be used to show that m will pass to the right of B if $m_1 < l_1$. Therefore m will pass through B if and only if $m_1 = l_1$ and $m_2 = l_2$ meaning $m = \ell$. □

Lemma 4.4.6. ***P6*** *holds in the Moulton Plane.*

Proof. The Moulton Plane can be coordinatised by the real numbers, a field of characteristic $\neq 2$, so by Theorem 4.3.10 it is a Fano Plane. □

Theorem 4.4.7 (Kadison and Kromann, Section 8.4 [10])**.** ***P6*** *and Desargues' Theorem are independent of one another.*

Proof. The Moulton plane satisfies **P6** but is non-Desarguesian. A projective plane over a division ring is a Desarguesian plane which does not satisfy **P6.**. Therefore neither can imply the other meaning they are independent. □

Theorem 4.4.8 (Kadison and Kromann, Section 8.4 [10])**.** ***P6*** *and Pappus' Theorem are independent of one another.*

Proof. Since the Moulton plane is non-Desarguesian by Theorem 4.2.1 which says that all Pappian planes are Desarguesian, the Moulton Plane is also non-Pappian. Therefore in the Moulton plane **P6** holds but not Pappus' Theorem. Meanwhile, Pappus' Theorem holds in the Fano Plane but **P6** does not. Again, neither can imply the other meaning the are independent. □

Chapter 5

Bricard's Theorem

5.1 Bricard from Desargues

We now introduce a new geometric proposition which is the focus of this thesis. The proposition is known as Bricard's Theorem, as it was originally studied in the real plane where it is always true. It is attributed to Raoul Bricard and traditionally thought to appear first in his book Géométrie Descriptive, published in 1911 [2], although several searches by myself and my supervisors have failed to find it. However, all works citing Bricard's Theorem still reference this book, so we shall too. In fact, there has been very little investigation of Bricard's Theorem since it was published and it appears in very few papers. It seems that everywhere it appears, no assumptions are given about the underlying structure of the projective plane. Is it $\mathsf{PG}(2, D)$? Is it $\mathsf{PG}(2, F)$? Is it a direct consequence of **P1**-**P4** and therefore holds in every projective plane? Only Bamberg and Penttila clarify they are working in $\mathsf{PG}(2, D)$ when giving a proof of Bricard's Theorem (private communication, 2021). The theorem is given in two directions $\mathrm{B}^{\mathbf{I}}$ and $\mathrm{B}^{\mathbf{II}}$ as we cannot assume they are equivalent. Bamberg and Penttila show one direction, we follow their approach to prove both directions.

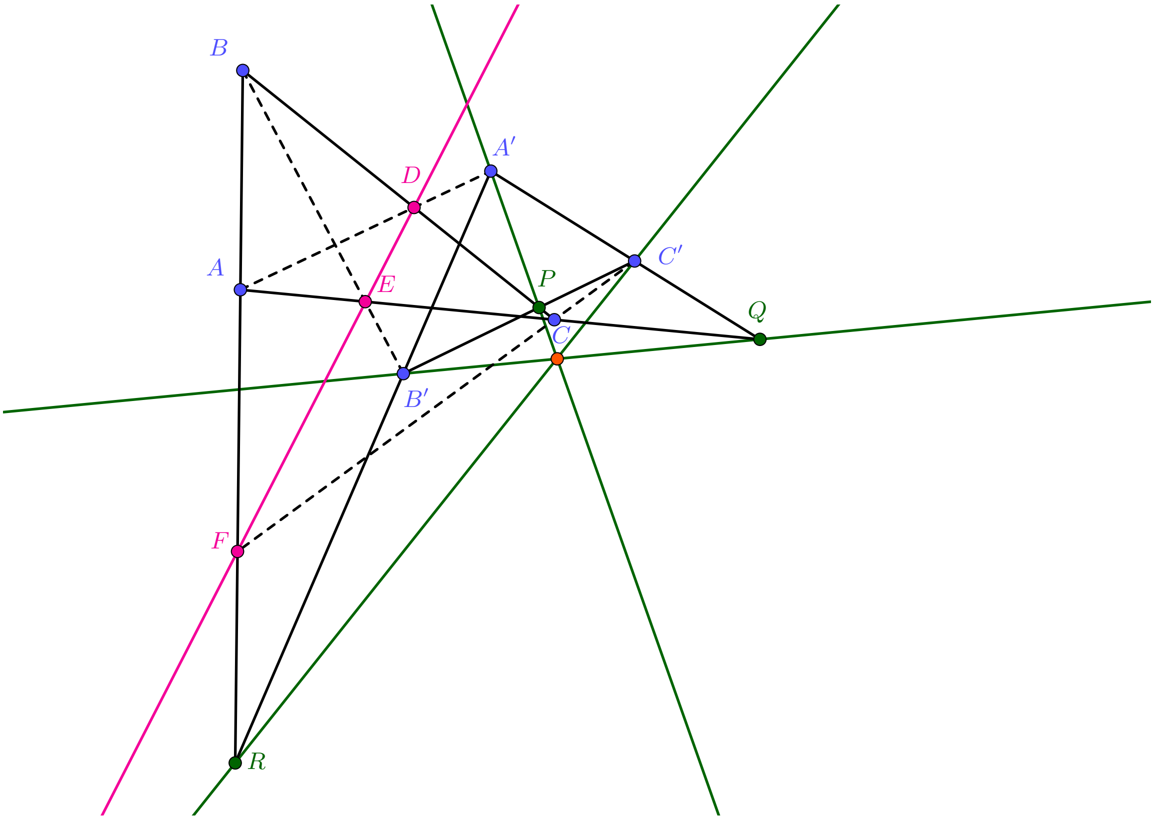


Figure 5.1: Bricard's Theorem

Theorem 5.1.1 ($\mathrm{B}^{\mathbf{I}}$)**.** *Let A, B, C, and A', B', C' be two triangles in a projective plane and let $P = BC \cap B'C'$, $Q = AC \cap A'C'$ and $R = AB \cap A'B'$. If the lines $A'P$, $B'Q$ and $C'R$ are concurrent, then the points $D = BC \cap AA'$, $E = AC \cap BB'$ and $F = AB \cap CC'$ are collinear.*

Theorem 5.1.2 ($\mathrm{B}^{\mathbf{II}}$)**.** *Let A, B, C, and A', B', C' be two triangles in a projective plane such that the points $D = BC \cap AA'$, $E = AC \cap BB'$ and $F = AB \cap CC'$ are collinear. Then the lines $A'P$, $B'Q$ and $C'R$ are concurrent, where $P = BC \cap B'C'$, $Q = AC \cap A'C'$ and $R = AB \cap A'B'$.*

Theorem 5.1.3. *Desargues' Theorem implies $B^{\mathbf{I}}$ and $B^{\mathbf{II}}$.*

The following construction is used to prove both theorems. $(3, D)$ acts transitively on triangles, so without loss of generality we are free to choose points $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. Firstly BC is the line $x = 0$, AC is the line $y = 0$ and AB is the line $z = 0$.

Let the line through DF be $[1, 1, 1]$. $D = (d_1, d_2, d_3)$ is on $[1, 1, 1]$ and $[1, 0, 0]$ so

$$d_1 + d_2 + d_3 = 0 \text{ and } d_1 = 0$$
$$\Rightarrow D = (0, -1, 1).$$

Similarly, $F = (f_1, f_2, f_3)$ is on $[1, 1, 1]$ and $[0, 0, 1]$ so

$$f_1 + f_2 + f_3 = 0 \text{ and } f_3 = 0$$
$$\Rightarrow F = (-1, 1, 0).$$

Now that we have the points D and F, we can calculate the lines $AD = [AD|_1, AD|_2, AD|_3]$ and $CF = [CF|_1, CF|_2, CF|_3]$.

$$AD|_1(1) = 0 \qquad (A \text{ on } AD)$$
$$AD|_2(-1) + AD|_3(1) = 0 \qquad (D \text{ on } AD)$$
$$\Rightarrow AD = [0, 1, 1].$$

$$CF|_3(1) = 0 \qquad (C \text{ on } CF)$$
$$CF|_1(-1) + CF|_2(1) = 0 \qquad (F \text{ on } CF)$$
$$\Rightarrow CF = [1, 1, 0].$$

Since A' is a point on AD not equal to D, we can say $A' = (\alpha, 1, -1)$ for some α. In the same way C' is a point on CF not equal to C so $C' = (1, -1, \gamma)$ for some γ. We can now calculate the line $A'C' = [A'C'|_1, A'C'|_2, A'C'|_3]$.

(3) $$0 = A'C'|_1(\alpha) + A'C'|_2(1) + A'C'|_3(-1) \qquad (A' \text{ on } A'C')$$

(4) $$= A'C'|_1(1) + A'C'|_2(-1) + A'C'|_3(\gamma) \qquad (C' \text{ on } A'C')$$

$$= A'C'|_1 + A'C'|_1(\alpha) + A'C'|_3(\gamma) - A'C'|_3 \qquad (\text{add 3 and 4})$$

(5) $$= A'C'|_1(\alpha + 1) + A'C'|_3(\gamma - 1)$$

Recall the definition of $Q = AC \cap A'C'$, so $Q = (q_1, q_2, q_3)$ is the intersection of $A'C'$ and $[0, 1, 0]$,

$$q_2 = 0$$
$$\Rightarrow A'C'|_1(q_1) + A'C'|_3(q_3) = 0$$
$$\Rightarrow Q = (\alpha + 1, 0, \gamma - 1) \qquad \text{(by 5)}$$

We cannot yet find R or P explicitly but we know that R lies on AB, which is the line $z = 0$, so we can say R is of the form $(r_1, r_2, 0)$, and P lies on BC which is the line $x = 0$ so P must be of the form $(0, p_2, p_3)$.Now the line $A'R$ has:

$$0 = A'R|_1\alpha + A'R|_2 - A'R|_3 \qquad (A' \text{ on } A'R) \tag{6}$$
$$= A'R|_1 r_1 + A'R|_2 r_2 \qquad (R \text{ on } A'R) \tag{7}$$
$$= A'R|_1(r_1 - \alpha) + A'R|_2(r_2 - 1) + A'R|_3 \qquad \text{(7 subtract 6)} \tag{8}$$

By the definition of R, B' lies on $A'R$ so by (8) we can say the $B' = (r_1 - \alpha, \beta, 1)$, where $\beta = r_2 - 1$, which also implies that $r_2 = \beta + 1$ which gives $R = (r_1, \beta + 1, 0)$. Lastly, we need to find the point P using the fact that P is the intersection of BC and $B'C'$. The line $B'C' = [B'C'|_1, B'C'|_2, B'C'|_3]$ has:

$$0 = B'C'|_1(r_1 - \alpha) + B'C'|_2(\beta) + B'C'|_3(1) \qquad (B' \text{ on } B'C') \tag{9}$$
$$= B'C'|_1(1) + B'C'|_2(-1) + B'C'|_3(\gamma) \qquad (C' \text{ on } B'C') \tag{10}$$
$$= B'C'|_1(r_1 - \alpha + 1) - B'C'|_2(\beta - 1) + B'C'|_3(\gamma + 1) \qquad \text{(add 9 and 10)}$$

So P is the intersection of this line with $x = 0$. We now write equations for the lines $A'P, B'Q$ and $C'R$. $A'P = [A'P|_1, A'P|_2, A'P|_3]$ has:

$$0 = A'P|_1(\alpha) + A'P|_2(1) + A'P|_3(-1) \qquad (A' \text{ on } A'P) \tag{11}$$
$$= A'P|_2(p_2) + A'P|_3(p_3) \qquad (P \text{ on } A'P) \tag{12}$$
$$= A'P|_1\alpha + A'P|_2(1 + p_2) + A'P|_3(p_3 - 1) \qquad \text{(add 11 and 12)}$$

$B'Q = [B'Q|_1, B'Q|_2, B'Q|_3]$ has:

$$0 = B'Q|_1(\alpha + 1) + B'Q|_3(\gamma - 1) \quad (13) \qquad (Q \text{ on } B'Q)$$

$$= B'Q|_1(r_1 - \alpha) + B'Q|_2\beta + B'Q|_3(1) \quad (14) \qquad (B' \text{ on } B'Q)$$

$$= B'Q|_1(r_1 + 1) + B'Q|_2\beta + B'Q|_3\gamma \qquad (\text{add 13 and 14})$$

$C'R = [C'R|_1, C'R|_2, C'R|_3]$ has:

$$0 = C'R|_1(1) + C'R|_2(-1) + C'R|_3(\gamma) \quad (15) \qquad (C' \text{ on } C'R)$$

$$= C'R|_1(r_1) + C'R|_2(\beta + 1) \quad (16) \qquad (R \text{ on } C'R)$$

$$= C'R|_1(r_1 + 1) + C'R|_2\beta + C'R|_3\gamma \qquad (\text{add 15 and 16})$$

A′P: $A'P|_1\alpha + A'P|_2(1 + p_2) + A'P|_3(p_3 - 1) = 0$

B′Q: $B'Q|_1(r_1 + 1) + B'Q|_2\beta + B'Q|_3\gamma = 0$

C′R: $C'R|_1(r_1 + 1) + C'R|_2\beta + C'R|_3\gamma = 0$

Proof. ($\mathrm{B}^{\mathbf{I}}$) If these three lines are concurrent then $A'P$ must pass through $(r_1 + 1, \beta, \gamma)$, the unique point of intersection of $B'Q$ and $C'R$. Therefore $\alpha = r_1 + 1$ meaning $r_1 = \alpha - 1$ which leads to $B' = (-1, \beta, 1)$.Now the line BB' has

$$0 = BB'|_1(-1) + BB'|_2\beta + BB'|_3(1) \qquad (B' \text{ on } BB')$$

$$0 = BB'|_2 \qquad (B \text{ on } BB')$$

$$\Rightarrow BB' = [1, 0, 1]$$

So $E = (e_1, e_2, e_3)$, which is the intersection of BB' and AC, is on $[1, 0, 1]$ and $y = 0$, so

$$e_1 + e_3 = 0 \text{ and } e_2 = 0$$

$$\Rightarrow E = (-1, 0, 1).$$

Lastly $1(-1) + 1(0) + 1(1) = 0$ so E must be on the line $[1, 1, 1]$, meaning D, E and F are collinear. □

Proof. ($\mathrm{B}^{\mathbf{II}}$) If DE and F are collinear, then E lies on $[1, 1, 1]$,

If $E = (e_1, e_2, e_3)$ lies on $[1, 1, 1]$ and $[0, 1, 0]$ we get that:

$$e_1 + e_2 + e_3 = 0 \text{ and } e_2 = 0$$
$$\Rightarrow E = (-1, 0, 1).$$

So the line $BE = [BE|_1, BE|_2, BE|_3]$ has

$$BE|_2(1) = 0 \qquad (B \text{ on } BE)$$
$$BE|_1(-1) + BE|_3(1) = 0 \qquad (E \text{ on } BE)$$
$$\Rightarrow BE = [1, 0, 1].$$

Using the same argument as before, B' is a point of BE not equal to B so $B' = (-1, \beta, 1)$. Earlier we said that $B = (r_1-\alpha, \beta, 1)$ which implies that $r_1-\alpha = -1$ therefore $r_1 = \alpha - 1$. Recall that P is the intersection of $B'C'$ and $[1, 0, 0]$,

$$\textbf{B'C':}\ 0 = B'C'|_1(r_1 - \alpha + 1) - B'C'|_2(\beta - 1) + B'C'|_3(\gamma + 1)$$
$$(17) \qquad = B'C'|_1(0) - B'C'|_2(\beta - 1) + B'C'|_3(\gamma + 1) \qquad (\text{sub in } r_1)$$

$$\Rightarrow P = (0, \beta - 1, \gamma + 1) \qquad (\text{by } 17)$$

We rewrite the equations for $A'P, B'Q$ and $C'R$ in terms of α, β and γ:

$$\mathbf{A'P:}\ A'P|_1\alpha + A'P|_2\beta + A'P|_3\gamma = 0$$
$$\mathbf{B'Q:}\ B'Q|_1\alpha + B'Q|_2\beta + B'Q|_3\gamma = 0$$
$$\mathbf{C'R:}\ C'R|_1\alpha + C'R|_2\beta + C'R|_3\gamma = 0$$

And it is clear to see that these three lines are concurrent in (α, β, γ).
This shows that Bricard's Theorem holds in all projective planes defined over a division ring. We can attribute this result to Bamberg and Penttila (private communication, 2021). However, we showed with Theorem 4.3.1 that all projective planes defined over a division ring are Desarguesian this means that Desargues' Theorem also implies Bricard's Theorem. □

5.2 Bricard in the Moulton Plane

We now turn our attention to the converse, i.e., Bricard's Theorem implies Desargues. One thing to note about the proof of $B^{\mathbf{I}}$ and $B^{\mathbf{II}}$ given in the previous section is that it doesn't appear to need associativity. There are no instances where three things are being multiplied together. We have already seen that Desargues' Theorem is the geometric equivalent of associative multiplication. Does this mean that $B^{\mathbf{I}}$ and $B^{\mathbf{II}}$ are weaker and do they hold in projective planes where Desargues' Theorem does not?

One quick way to check this would be to see if either direction of Bricard's Theorem holds in the Moulton Plane, which we know is non-Desarguesian. If either $B^{\mathbf{I}}$ or $B^{\mathbf{II}}$ hold in the Moulton Plane then we know immediately that they cannot imply Desargues' Theorem. If they don't then we need to do further analysis. Before attempting to prove that $B^{\mathbf{I}}$ and $B^{\mathbf{II}}$ do hold in the Moulton Plane, first we will look for a counterexample of them failing. As it happens this is very easy, we simply follow the method used to construct an example of Desargues' Theorem failing in the Moulton Plane. The trick is to position the entire figure such that exactly one of the points lies below the x-axis and rotate it so that exactly one of the lines through that point has a positive gradient.

Proposition 5.2.1. *$B^{\mathbf{I}}$ and $B^{\mathbf{II}}$ fail in the Moulton Plane.*

Proof. Firstly $B^{\mathbf{II}}$, which says that collinearity implies concurrency. Take points $A = (11, 8)$, $B = (12, 12)$, $C = (-2, 10)$, $A' = (0, 14)$, and $B' = (4, 6)$. If DEF are collinear, then we are free to let C' be any point on CF. Choose $C' = (-6, 11.6)$. Recall R is the intersection of AB and $A'B'$. In the real plane this would be the intersection of $y = 4x-36$ and $y = -2x+14$ which is $(8.3, -2.7)$, call this point R_{real}.

However, in the Moulton Plane the line AB is a line with positive gradient therefore it is refracted by a factor of 2 as it crosses the x-axis. The x-intercept is

$(9,0)$ so the portion of AB located below the x-axis is the line with gradient 8 which has an x-intercept at $(9,0)$ also. This is the line $y = 8x - 72$. So, the point R is the intersection of $y = -2x + 14$ and $y = 8x - 72$ which is $(8.6, -3.2)$.

Since Bricard's Theorem holds in the real projective plane, the lines $A'P$, $B'Q$ and $C'R_{real}$ are concurrent. It is easy to calculate that the point R is not on $C'R_{real}$ which means that $A'P$, $B'Q$ and $C'R$ cannot be concurrent, in other words $\mathrm{B}^{\mathbf{II}}$ fails in the Moulton Plane. This can be seen easily in Figure 5.3.

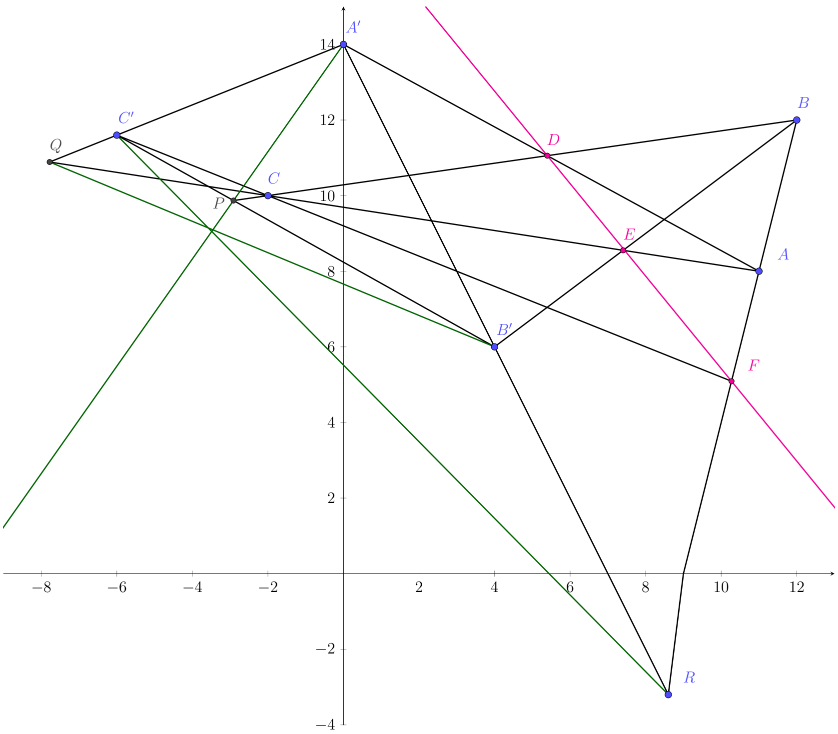


Figure 5.2: $\mathrm{B}^{\mathbf{II}}$ fails in the Moulton Plane

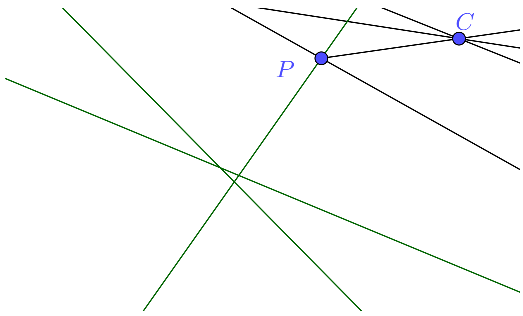


Figure 5.3: Close up of Figure 5.2

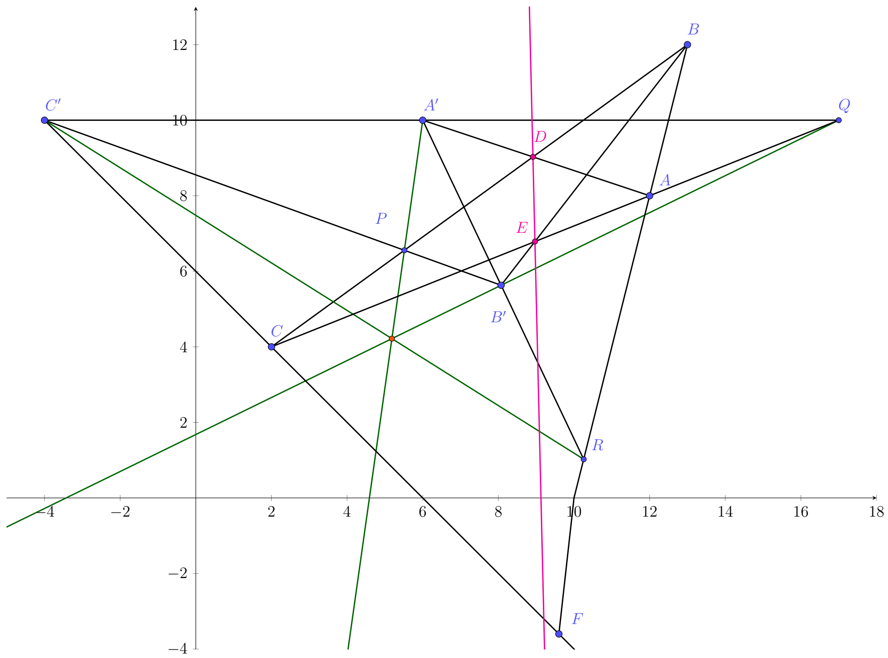


Figure 5.4: $\mathrm{B}^{\mathbf{I}}$ fails in the Moulton Plane

Next we consider $\mathrm{B}^{\mathbf{I}}$, which says that concurrency implies collinearity. This time take $A = (12, 8)$, $B = (13, 12)$, $C = (2, 4)$, $A' = (6, 10)$ and $C' = (-4, 10)$. In order for $A'P$, $B'Q$ and $C'R$ to be concurrent we must have $B' = (8.1, 5.6)$. Now, recall the F is the intersection of AB with CC'. In the real plane this would be the intersection of $y = 4x - 40$ with $y = -x + 6$ which is $(9.2, -3.2)$.

However, in the Moulton Plane AB is a a line with positive gradient meaning it is refracted by a factor of 2 as it crosses the x-axis, which is at the point $(10, 0)$. Therefore, the portion of AB located below the x-axis is the line with gradient 8 and x-intercept $(10, 0)$ which is the line $y = 8x - 80$. So in the Moulton Plane, the point F is the intersection of $y = -x + 6$ with $y = 8x - 80$ which means $F = (9.6, -3.6)$.

From here we can calculate the position of D and E and the line which joins them. It is then easy to show that F is not on this line, meaning D, E and F are not collinear or in other words $\mathrm{B}^{\mathbf{I}}$ fails in the Moulton Plane. This configuration is given in Figure 5.4.

$\square$

5.3 Bricard Summary

To summarise what we know about Bricard's Theorem so far, we have seen that:

1. Desargues' Theorem implies both $B^{\mathbf{I}}$ and $B^{\mathbf{II}}$.

2. As a result of point **1.** $B^{\mathbf{I}}$ and $B^{\mathbf{II}}$ are equivalent in a Desarguesian plane.

3. Bricard's Theorem doesn't require associative multiplication, meaning it could be weaker than Desargues' Theorem.

4. Bricard's Theorem fails in the Moulton Plane. We cannot use the Moulton Plane as a counterexample to show Bricard does not imply Desargues.

These are all useful bits of information however we are still unable to determine whether or not either $B^{\mathbf{I}}$ or $B^{\mathbf{II}}$ imply Desargues' Theorem. In fact, we still don't know if they are always equivalent to one another, or only in a Desarguesian Plane.

In order to introduce a potential method of answering these questions, we turn our attention once again to algebra. For example what is the exact algebraic significance of Bricard's Theorem and what happens in the finite case? Knowing the answer to the first of these questions will almost certainly help to answer the second. We suspect Bricard's Theorem does not require associative multiplication which suggests it is not equivalent to the full version of Desargues' Theorem. What is Desargues' Theorem without associative multiplication? Our work on algebraic consequences tells us that the next most likely candidate is D_{10}, which is equivalent to an alternative division ring, a division ring without associative multiplication. This could mean for example that Bricard is actually equivalent to D_{10}. If so the picture would look like this:

Suppose there was an arrow from Bricard to Desargues. Since there is already an arrow from Bricard to D_{10} then this would imply another arrow from D_{10} to Desargues, however we can be certain that D_{10} does not imply D_{11} because of the octonions. This would form a contradiction and be one way to show Bricard cannot imply Desargues, allowing us to answer our original question.

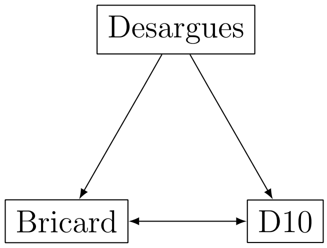


Figure 5.5: Potential Relationship between Bricard and D_{10}

Having a geometric proof of how Desargues' Theorem implies Bricard would make it easy to see whether either of the two triangles used in D_{11} had a vertex lying on the side of the other which would reveal whether D_{10} also implies Bricard. Then we would be well on our way to utilising the method outlined above.

One last question to ask is what is the dual of Bricard, and is Bricard equivalent to its dual?

Chapter 6

Trilinear Polar

6.1 Trilinear Polar from Bricard

We ended the previous chapter by signalling the importance of direct geometric proofs. However if we are looking for applications of any of these theorems in others, one important observation to make is that the configuration of Desargues' Theorem involves 10 points and 10 lines while Bricard's Theorem involves 13 points and 13 lines. This makes the task more difficult as there can be no direct correlation between the two theorems.

Recall Pappus' Theorem and Desargues' Theorem both had degenerate versions involving extra incidences between points and lines. In the case of Bricard's theorem, perhaps we can do the same thing. Ideally introducing three extra incidences between points and other points, to reduce the number of points from 13 to 10. In this way, we have a better chance of finding a direct link to Desargues' Theorem or perhaps even one of D_{10} or D_9. This motivates the following corollary of Bricard's Theorem.

Theorem 6.1.1 (Trilinear Polar). *Let A, B, C, and A', B', C' be two triangles in a projective plane such that the point A lies on $A'C'$, B lies on $A'B'$ and C lies on $B'C'$. Then the points $D = BC \cap AA'$, $E = AC \cap BB'$ and $F = AB \cap CC'$ are collinear if the lines $A'C$, $B'A$ and $C'B$.*

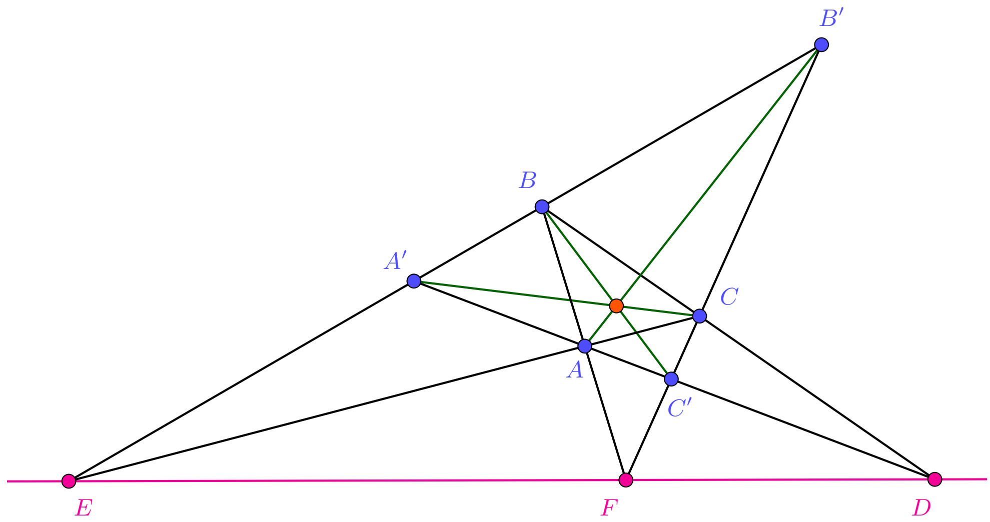


Figure 6.1: Trilinear Polar

As it happens this is precisely the statement given by the Trilinear Polar Theorem attributed to J.-J.-A. Mathieu in 1865 [13]. The requirement that A lies on $A'C'$, B lies on $A'B'$ and C lies on $B'C'$ means that $P = C$, $Q = A$ and $R = C$. With this we have reduced the number of points by three, meaning there are now 10 points remaining, the same number as in Desargues' Theorem. One possible configuration of Trilinear Polar is given in Figure 6.1 and already looks distinctly more Desarguesian than the full version of Bricard's Theorem, however it is still clear to see that Trilinear Polar is a direct result of Bricard's Theorem. As far as we can tell this connection has not been made before.

However, the Trilinear Polar Theorem is well known and arises in a number of other contexts. For example, another way to phrase the Trilinear Polar Theorem is that the harmonic conjugates of C with respect to $B'C'$, B with respect to $A'B'$ and A with respect to $A'C'$ are collinear. This statement can be proven using the cross ratio, $R(a, b, c, d)$ which is defined on a set of four collinear points and is invariant under projective transformations.

$$R(a, b, c, d) = \frac{a - c}{a - d} \cdot \frac{b - d}{b - c}$$

The cross ratio is a result of the theorems of Giovanni Ceva and Menelaus, which in the real projective plane are the dual of each other. Menelaus' Theorem first appears in his book *Spaerica* dating back to roughly 100 AD while Ceva published his theorem in 1678. However, although it is attributed to Ceva it appears to have been proven much earlier by Yusuf Al-Mu'taman ibn Hűd, an eleventh-century king of Zaragoza [8]. In algebraic terms, the cross ration implies we only need inverses to commute, similar to the case for D_9. However, there could also be other algebraic consequences which are a result of Trilinear Polar.

A third way to express Trilinear Polar theorem follows. However, first we need the definition of the Cevian triangle, named after Ceva, and the trilinear pole and polar, from which the theorem gets its name.

Definition 6.1.2 (Cevian Triangle)**.** Given a triangle ABC and a point P the **cevian triangle** is defined as the triangle with vertices $D = AP \cap BC$, $E = BP \cap AC$ and $F = CP \cap AB$. A triangle and its cevian are always in perspective from the point P which is referred to as the **trilinear pole** while the line through the intersections of corresponding sides is called the **trilinear polar**.

With this definition in mind we can can restate the Trilinear Polar theorem as follows: The intersections of the corresponding sides of a triangle and it's cevian are collinear. When phrased like this is is easy to see that Trilinear Polar is a direct consequence of Desargues Theorem (because the two triangles are always centrally perspective) but also D_{10} and D_9 as well. Trilinear Polar can also be thought of as D_8, or Desargues with three extra incidences.

6.2 Trilinear Polar in the Hierarchy

Theorem 6.2.1. *Special Desargues implies Trilinear Polar.*

Proof. Let ABC, and $A'B'C'$ be two triangles in a projective plane such that the point A lies on $A'C'$, B lies on $A'B'$ and C lies on $B'C'$ and the lines $A'C$, $B'A$ and

$C'B$ are concurrent. In other words they are centrally perspective from a point P. We want to show the points $D = BC \cap AA'$, $E = AC \cap BB'$ and $F = AB \cap CC'$ are collinear. Note that we have two triangles in perspective from a point such that one triangle has a vertex lying on a side of the other. So by Special Desargues the intersections of their corresponding sides are collinear. That is, $AB \cap B'C'$, $AC \cap A'C'$ and $BC \cap A'C'$ are concurrent. But C lies on $B'C'$ by definition so $B'C'$ is the same line as CC'. Using the same argument $A'C'$ is the same line as AA' and $A'B'$ is the same line as BB'. Now we have $BC \cap AA'$, $AC \cap BB'$ and $AB \cap CC'$ are collinear which is precisely the points D, E and F. □

Theorem 6.2.2. *D_9 implies Trilinear Polar.*

Remark. The proof is almost identical to the one above, but instead we have one triangle with two vertices lying on two sides of the other triangle, so we are able to use the proposition of D_9 to show that the intersections of corresponding sides are collinear.

Theorem 6.2.3. *Central Little Pappus plus **P6** implies Trilinear Polar.*

Proof. Given triangle ABC and a point P, construct the cevian triangle DEF where $D = AP \cap BC$, $E = BP \cap AC$ and $F = CP \cap AB$. Let $X = BC \cap EF$, $Y = AC \cap DF$ and $Z = AB \cap DE$. We want to show X, Y and Z are collinear. Consider lines AF and CD which meet at B. The intersection of CF and AD is the point P. The intersection of DZ and FX is E because X is on EF and Z is on ED. We also know that B, E and P are collinear because we constructed E to be that way. By Little Pappus' Theorem to show that $V = AX \cap CZ$ is also collinear with B, E and P. By the same argument $U = BY \cap CZ$ is collinear with A, P and D and $W = AX \cap BY$ is collinear with C, P and F.

Now consider the three collinear points A, C and Y, and the point V. The line UW is a line through Y which meets VC in U and VA in W. The intersection of

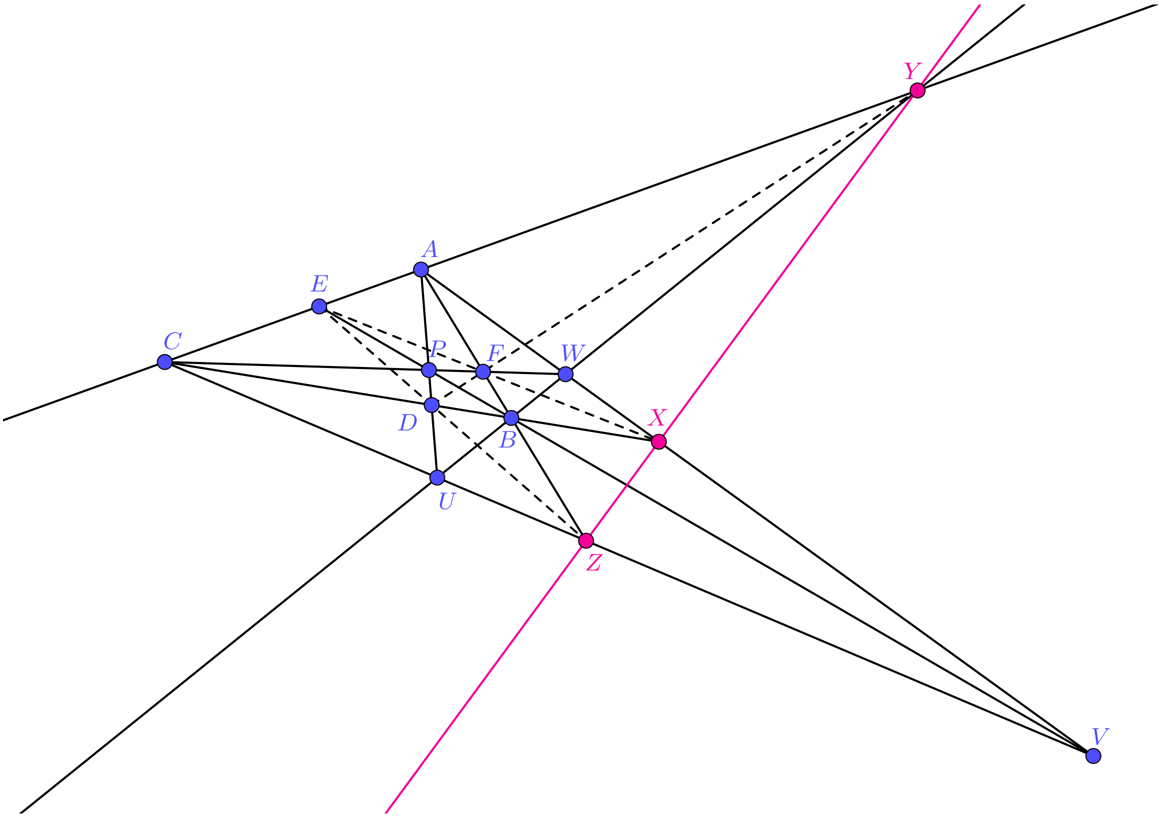


Figure 6.2: Little Pappus with **P6** implies Trilinear Polar

AU and CW is P because U is on AD and W is on CF. The intersection of VP with AC is E because V is on BP. So we have that E is the harmonic conjugate of Y with respect to A, C and conversely Y is the harmonic conjugate of E.

Next consider points X and Z. The line XZ meets VC in Z and VA in X. The intersection of ZA with CX is B because Z is on AB and X is on CB. Now $VB \cap AC$ is also E because B, P and V are collinear. Since the harmonic conjugate is independent of the quadrangle used to construct it, the intersection of XY with AC must produce the same point as UW with AC which is precisely the points Y. Therefore X, Y and Z must be collinear. □

As well as Central Little Pappus, this theorem also uses **P6**. Moufang thought that D_9 was equivalent to the harmonic conjugate being independent of the quadrangle used to construct it. As John Stillwell points out in his translation of her notes *Grundlagen der Geometrie* from 1948[14], this is a mistake. D_9 implies the harmonic conjugate is in fact independent of the quadrangle used to construct it however the converse is not true, that is if the harmonic conjugate is independent of the quadrangle used to construct it, D_9 does not necessarily hold. This is important as otherwise Little Pappus on its own would not imply trilinear polar, we would also be using D_9 as well. D_9 is actually equivalent to the harmonic conjugate being unique, which is a stronger property than **P6** or equivalently that the harmonic conjugate of a point is never equal to itself. The uniqueness of the harmonic conjugate is proven by Seidenberg in [21] while proof that D_9 is equivalent to the harmonic conjugate being unique can be found in Heyting, Theorem 2.4.3 [6].

Theorem 6.2.4. *D_9 implies $B^{\mathbf{I}}$ with two pairs of coincident points.*

Proof. Let H, I and J be three distinct points in a projective plane. Let A, B, C, A', B', and C' be distinct points such that **1)** $AA' \cap BB' = I$, $AA' \cap CC' = J$ and $BB' \cap CC' = H$ and **2)** $A'P$, $B'Q$ and $C'R$ are concurrent, where $P = BC \cap B'C'$, $Q = AC \cap A'C'$ and $R = AB \cap A'B'$. We want to show the points $D = BC \cap AA'$, $E = AC \cap BB'$ and $F = AB \cap CC'$ are collinear.

The first condition implies that $AA' = IJ$, $BB' = HI$ and $CC' = HJ$. Now we have that $D = BC \cap IJ$, $E = AC \cap HI$ and $F = AB \cap HJ$. Assuming D_9 holds, Trilinear Polar also holds and D, E and F will be collinear if one of triangles ABC or HIJ is the cevian of the other. If $A' = I$, $B' = H$ and C' is any point on HJ, then

$$R = AB \cap A'B' = AB \cap HI = B$$

$$P = BC \cap B'C' = BC \cap HJ = C$$

Let $L = QP \cap A'B'$. The second condition means $A'P$, $B'Q$ and $C'R$ are concurrent, so R is the harmonic conjugate of L with respect to A', B'. Now D_9 is equivalent to the uniqueness of the harmonic conjugate. If we change the quadrangle we must still get the same harmonic conjugate. Recall $A' = I$ and $B' = H$, we can see that

$AC = QP$ because Q is on AC and $IA \cap HC = J$. Let $K = CI \cap AH$, RK must go through J, but $R = B$ so BK goes through J, meaning IC, HA and JB are concurrent. Also since A is on IJ, B is on HI and C is on HJ, ABC is the cevian of HIJ. □

Conjecture 1. *Neither $B^{\mathbf{I}}$ nor $B^{\mathbf{II}}$ imply Desargues.*

Remark. We have discovered evidence to suggest that Bricard's Theorem is actually far more closely related to Trilinear Polar than it is to Desargues' Theorem. In light of Theorem 6.2.4, we conjecture that it is equivalent to D_9, however we have not yet proven this equivalence without extra conditions on the points of Bricard's Theorem. Since it is known that D_{10} does not imply Desargues' Theorem, it follows that no theorem that sits beneath it in the hierarchy does either. Therefore if Bricard's Theorem is equivalent to any of the three listed above, it does not imply Desargues' Theorem.

Chapter 7

Concluding Remarks

Geometric propositions are a wonderful tool for classifying projective planes. Indeed some of them, including the theorems of Pappus and Menelaus have been around since long before the idea projective geometry was conceived. Although many of them can be depicted by a simple diagram consisting of 10 or so points and lines, we have discovered that the relationships between them are actually rather complex. We have provided a summary of the known relationships between Pappus' Theorem, Desargues' Theorem, Fano's Axiom and the Reidemeister condition. Beginning in 1905 with Hessenberg's incomplete proof, relationships between these theorems were explored throughout the 20th century. However, there remain some intriguing open questions regarding versions of these theorems involving extra incidences between points and lines. A list is complied below.

- D_9 is equivalent to D_{10} is a Fano Plane, but is this true in general?
- Are all possible configurations of Desargues' Theorem with two extra incidences equivalent?
- Does either P_9 or dP_9 imply D_9?
- Are P_9 and dP_9 equivalent?

In addition to the relationships between geometric propositions, we also gave a brief overview of the algebraic conditions on the planar ternary rings that are implied by them. A clear correspondence can be seen between the hierarchy of the geometric propositions and their underlying algebraic structures. This alludes to the fact that geometry and algebra are inherently entwined.

Lastly, we attempted to introduce Bricard's Theorem, another less well known geometric proposition from the early 20th century, into the classification. Initially motivated by the question 'Does Bricard's Theorem imply Desargues' Theorem, we uncovered the central role played by Trilinear Polar, yet another geometric proposition, published in 1865 but overlooked by both Moufang and Heyting.

It appears that Trilinear Polar is the weakest of the propositions and is implied by all of the others. An exploration of the relationship between Trilinear Polar and Bricard led us to prove that D_9 implies Bricard's Theorem in the case where two pairs of points from Bricard overlap (condition (1) in Figure 7.1), however we have also discovered strong evidence to suggest the condition can be removed and the implication will still hold. This leads us to the conjecture that Bricard's Theorem is equivalent to D_9. For this reason, we suspect Bricard's Theorem does not imply Desargues' Theorem because the octonions are a counterexample proving Desargues' Theorem is not equivalent to any weaker geometric proposition, however there is still a lot more research to be done on Bricard's Theorem.

Additional relationships between geometric properties presented in this thesis are summarised in Figure 7.1. In this case, **P6** is taken as an axiom of the projective plane to simplify the graph. Once again an arrow means 'implies', no arrow means 'does not imply' and a dotted arrow suggests it is still an open problem. Arrows in red indicate original work. Algebraic consequences are summarised in Figure 7.2, where the dotted rectangle represents the placement of Bricard as suggested by our research.

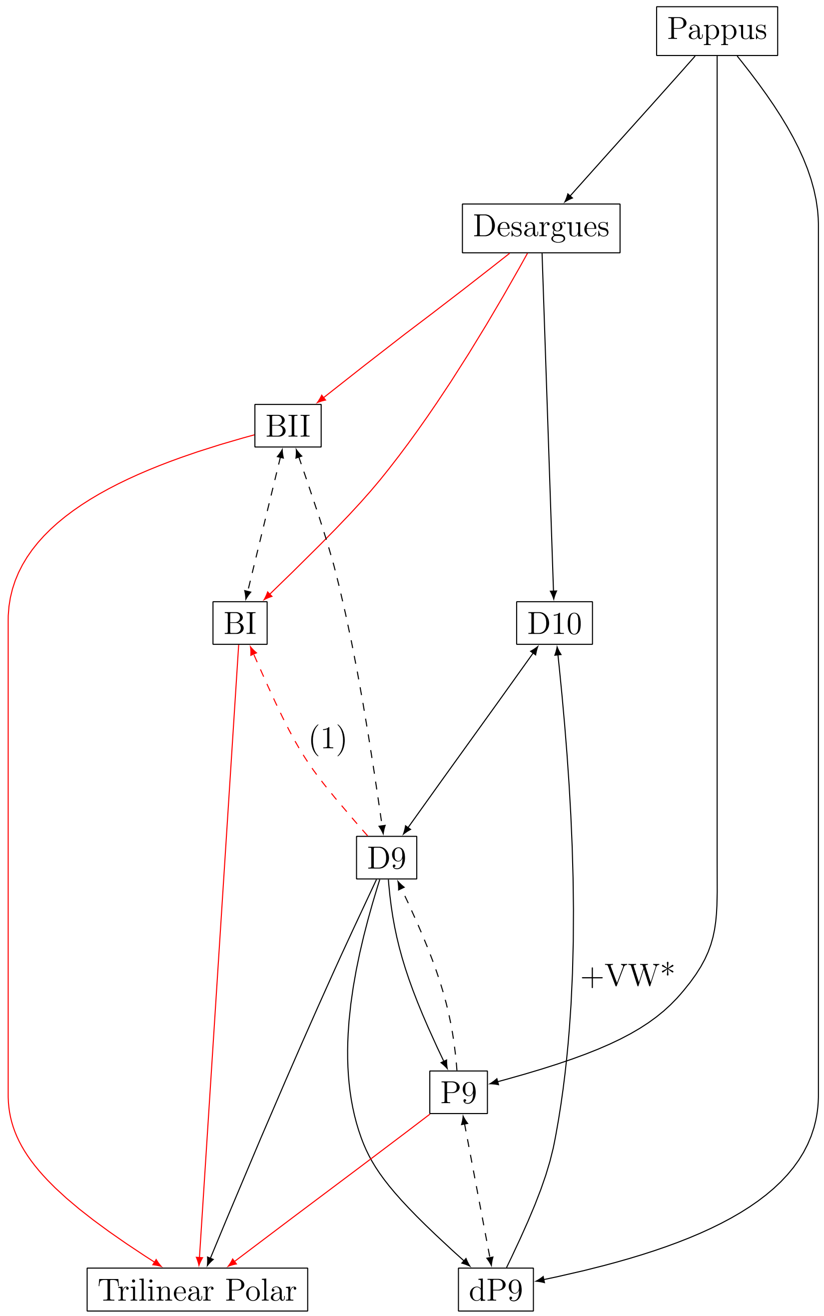


Figure 7.1: Summary of Relationships in a Fano Plane

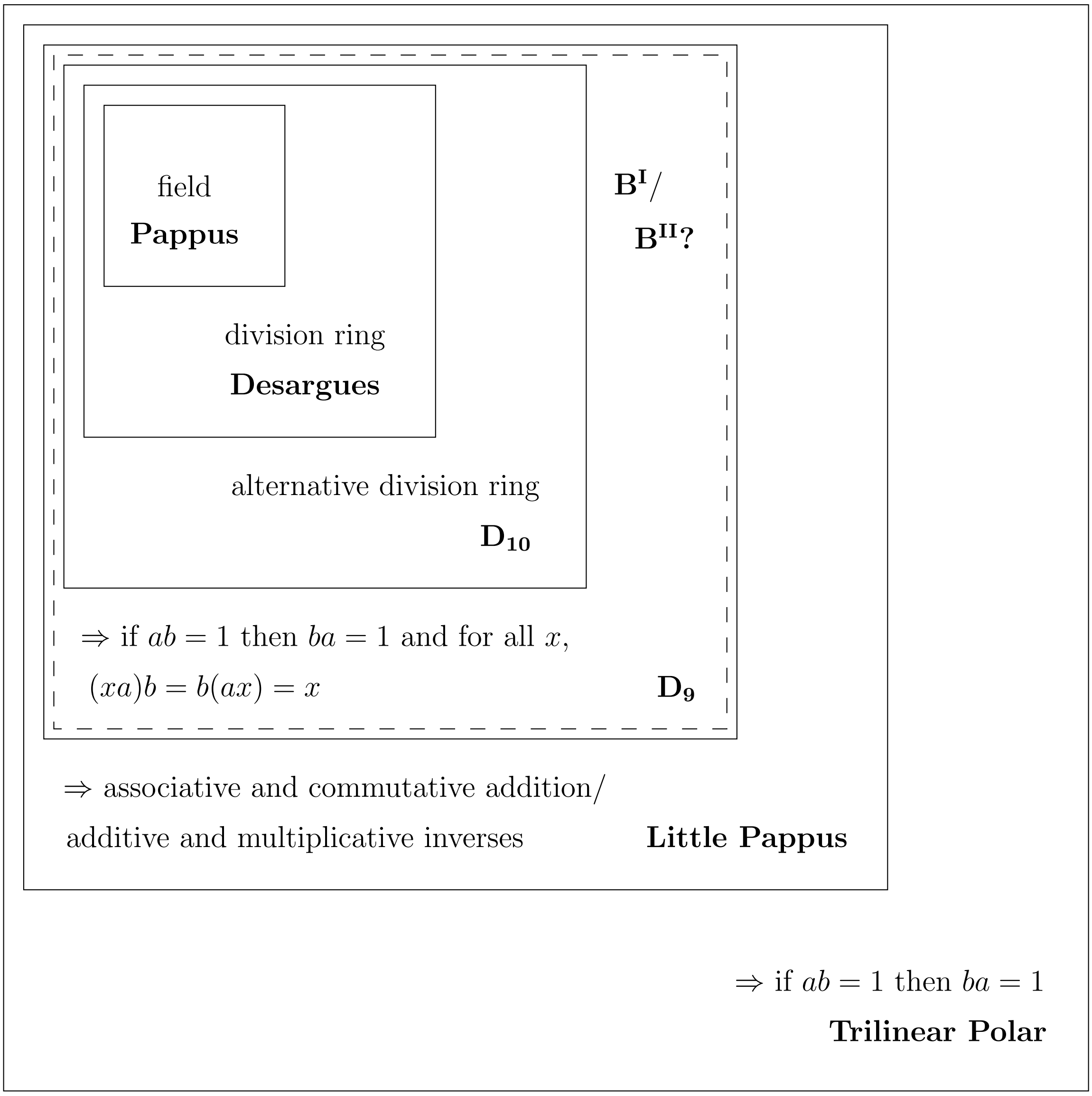


Figure 7.2: Summary of algebraic consequences

Bibliography

[1] A. Bosse. *Manière universelle de Mr. Desargues pour pratiquer la perspective.* 1648.

[2] R. Bricard. *Géométrie Descriptive.* Octave Doin et Fils, 1911.

[3] L. Da Vinci and H. Suh. *Leonardo's Notebooks: Writing and Art of the Great Master.* Notebook Series. Running Press, 2013.

[4] M. Hall. Projective planes. *Trans. Amer. Math. Soc.*, 54:229–277, 1943.

[5] M. Hall, J. Swift, and R. J. Walker. Uniqueness of the projective plane of order eight. *Mathematics of Computation*, 10:186–194, 1956.

[6] A. A. Heyting. *Axiomatic projective geometry.* Bibliotheca mathematica, a series of monographs on pure and applied mathematics ; v. 5. P. Noordhoff, Groningen, 2d ed. edition, 1980.

[7] D. Hilbert. *Grundlagen der Geometrie.* Vieweg+Teubner Verlag, 13 edition, 1968.

[8] A. Holme. *Geometry Our Cultural Heritage.* Springer Berlin Heidelberg, Berlin, Heidelberg, 2nd ed. 2010. edition, 2010.

[9] D. Hughes and F. Piper. *Projective Planes.* Graduate texts in mathematics. Springer-Verlag, 1973.

[10] L. Kadison. *Projective geometry and modern algebra.* Birkhauser, Boston, 1996.

[11] M. Kallaher. Reidemeister projective planes. 1967.

[12] H. Lüneburg. Uber die beiden kleinen satze von pappos. *Archiv der Mathematik*, 11:339–341, 1960.

[13] J.-J.-A. Mathieu. Etude de géométrie comparée, avec applications aux sections coniques. *Nouvelles annales de mathématiques : journal des candidats aux écoles polytechnique et normale*, pages 529–537, 1865.

[14] R. Moufang and J. Stillwell. Translation of moufang's'grundlagen der geometrie', 2020.

[15] F. R. Moulton. A simple non-desarguesian plane geometry. *Transactions of the American Mathematical Society*, 3:192–195, 1902.

[16] Pappus of Alexandria. *Pappus of Alexandria Book 7 of the Collection*. Springer-Verlag New York, 1 edition, 1986.

[17] G. Pickert. *Projektive Ebenen*. Springer, Berlin, Heidelberg, 2 edition, 1975.

[18] J.-V. Poncelet. *Traité des propriétés projectives des figures*. L'école Impériale Polytechnique, 2 edition, 1865.

[19] K. Reidemeister. *Topologische Fragen der Differentialgeometrie. V. Gewebe und Gruppen*, volume 29. 1929.

[20] A. Seidenberg. Pappus implies desargues. *The American Mathematical Monthly*, 83(3):190–192, 1976.

[21] A. A. Seidenberg. *Lectures in projective geometry*. The University series in undergraduate mathematics. Van Nostrand, Princeton, N.J, 1962.

[22] J. Stillwell. *The four pillars of geometry*. Undergraduate texts in mathematics. Springer, New York, 2005.

[23] O. Veblen and J. Young. *Projective Geometry*. Number v. 1 in Projective Geometry. Ginn, 1910.