Skew Projection

by

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Abstract

In this thesis, we investigate skew projection. More specifically, we are interested in determining how skew projection generates a conic. We give an introduction to the history of the problem, including some history of geometry in the 19th Century and the influence of notable geometers Jakob Steiner and Jean-Victor Poncelet. We also provide relevant background theory to the problem, including an overview of projective geometry, conics, forms, reguli, quadric surfaces, and group actions.

We then proceed to the main results. We provide a rigorous proof of the existence of skew projection. We then prove the existence of a hyperbolic quadric generated by three skew lines and their regulus by proving that the action of $\mathrm{PGL}(3, \mathbb{R})$ on triples of skew lines is transitive. We then classify projective quadrics via Sylvester's Law of Inertia and in addition to the main results, we classify the affine quadrics. We then determine the orbits of the hyperbolic quadric on degenerate and non-degenerate planes via Sylvester's Law and Witt's (Extension) Theorem. By determining these orbits, we ascertain the different conics generated by skew projection and determine the conditions on planes and on lines for generating these conics.

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Benedic, anima mea, Domino: et omnia, quæ intra me sunt, nomini sancto eius.
Benedic, anima mea, Domino: et noli oblivisci omnes retributiones eius.

— Ps 103 (102):1-2

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May God bless you,

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A.M.D.G. & B.V.M. ✝

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Chapter 1

Introduction

In this thesis, we will investigate skew projection. More specifically, we are interested in determining how skew projection generates a conic. In Chapter 1, we will provide an introduction to the history of the problem, including some history of geometry in the 19th Century and the influence of notable geometers Jakob Steiner and Jean-Victor Poncelet. We will also briefly introduce the reader to skew projection and explain the set-up of the problem in further detail in Section 1.3. In Chapter 2, we will provide relevant background theory to the problem, including an overview of projective geometry, conics, forms, reguli, quadric surfaces, and group actions.

We then proceed to the main results in Chapter 3. We will provide a rigorous proof of the existence of skew projection and the existence of a hyperbolic quadric generated by three skew lines and their regulus. We will achieve the latter by proving that the action of $\mathrm{PGL}(3,\mathbb{R})$ on triples of skew lines is transitive. We will then classify projective quadrics via Sylvester's Law of Inertia and we classify the affine quadrics. After which, we will determine the orbits of the stabiliser of the hyperbolic quadric on degenerate and non-degenerate planes via Sylvester's Law and Witt's (Extension) Theorem. By determining these orbits, we will ascertain the different conics generated by skew projection. In particular, we will determine the conditions on planes and on lines for generating these conics.

1.1 Geometry in the 19th Century

After having its naissance – to the best of our knowledge – with the Ancients in Greece, being taken up by the Arabic and Persian mathematicians a few millennia afterwards, geometry found its way to modern Europe somewhere around the 16th Century [7]. During the 19th Century, European geometry flourished. It was during this century that the modern fields of projective and algebraic geometry were established [11].

This thesis will investigate *skew projection*, a concept thought to have been

Figure 1.1: Jakob Steiner (1796-1863) [16].

first formalised by Swiss-German mathematician Jakob Steiner in the 19th Century. Contributions to skew projection were also made by French mathematician Jean-Victor Poncelet. Both mathematicians led interesting lives and made great contributions to 19th Century geometry.

Jakob Steiner

Jakob Steiner (1796-1863) was a Swiss mathematician, 'the man who has been regarded as the greatest synthetic geometer of modern times' [7]. His biographies often attribute to him a great zeal for geometry; he is thought to have said, 'Calculation replaces, whilst geometry stimulates, thinking' [16]. Steiner's most notable work was published in 1832, entitled *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*, which can be roughly translated as *The systematic development of the dependence of geometric shapes on one another* (henceforth, we will refer to the work as *geometrischer Gestalten*) [25]. It is viewed as a seminal work in establishing the newly emerging field of projective geometry (it was in this work that skew projection was introduced). A new chair of geometry was created for him in Berlin in 1834, which he held until his death. One of his most notable theorems, also attributed to Poncelet, is the so-called Poncelet-Steiner Theorem:

> All Euclidean geometric constructions can be carried out with a straightedge alone if, in addition, one is given the radius of a single circle and its [centre].

Jean-Victor Poncelet

Meanwhile in France, Jean-Victor Poncelet (1788-1867) was approaching geometry from a different angle. Poncelet studied under another notable French mathem-

Figure 1.2: Jean-Victor Poncelet (1788-1867) [15].

atician, Gaspard Monge, and after graduation, he joined the Corps of Military Engineers and served under Napoleon in the Russian invasion of 1812 [15]. This was, of course, a failure, for it is well known that one should never invade Russia during winter. Poncelet was taken prisoner by the Russians and whilst in prison wrote his own seminal work on projective geometry, entitled *Traité des propriétés projectives des figures*, which translates to *Treatise on the projective properties of figures* (henceforth, we will refer to the work as *Traité*) [19, 20]. Though written during the years 1812-1814, the work could not be published until his release in 1814, after which he wrote one, perhaps two, more volumes of the *Traité*; hence the publication date is usually given much later as 1865. The *Traité* is seen as a foundational work for projective geometry, drawing together the prevailing knowledge of the time. Some see Poncelet as 'the effective founder of projective geometry' [7].

1.2 History of the problem

As it appears in the works of Steiner and Poncelet

The link between Steiner, Poncelet, and skew projection appears to have come from an online biography of Steiner, in which the following is stated:

> Proposition 59 [of Steiner's *geometrischer Gestalten*], labelled 'general observation', contains the 'skew projection', a quadratic relationship in space, sometimes called the 'Steiner relationship', which had been noted by Poncelet.

The proposition alluded to is a section in *geometrischer Gestalten* devoted to the hyperboloid and its properties, as stated in the opening line of Proposition 59 (emphasis added):

> Das einfache **Hyperboloïd** giebt, vermöge der ihm zukommenden Eigenschaften und namentlich wermöge seiner doppelten Erzeugung durch projectivische Gebilde, ein Mittel an die Hand die gegenseitige Abhahängigkeit gewisser Systeme verschiedenartiger Figuren von einander klar darzutbun, die Uebertragung der Eigenschaften jedes Systems auf alle übrigen leicht zu bewerkstelligen, und zugleich auch jedes System in jedes audere zu verwandeln.

Unfortunately, it is difficult to obtain translations of *geometrischer Gestalten* and available copies of the work are image scans, making it difficult to perform rough translations of the work. However, the focus on hyperboloids was enough to realise the link between quadrics and skew projection, which became integral to this thesis.

Furthermore, the online biography provides a reference to Poncelet's *Traité*, Section III, Chapter II. This reference is elusive, for two reasons. Firstly, there are two volumes of Poncelet's *Traité*. Secondly, this chapter, in either of the volumes, has no clear relation to the problem. A few oblique references are made to the problem in other sections of the work, including one in the supplementary chapter of the first volume ('Supplément sur les propriétés projectives des figures dans l'espace', which translates to 'Supplement on the projective properties of figures in space'). In this chapter, Article 581 poses a question about skew lines and transversals. There are also references to quadric hyperboloids in the second volume, including one which appears to relate the generation of a hyperboloid by a regulus and its opposite regulus. The formulation of skew projection as it appears in this thesis does not appear to be included in the *Traité* and neither do any of the statements mentioned from the *Traité* go into further detail or proofs.

As it appears in the literature

Apart from Steiner and Poncelet, the problem is not widely represented in the literature: only two references to the formulation of skew projection used in this thesis could be found. In 1959, the problem was included in a mathematical reference work [23], in which the author argues that this seemingly simple observation on lines in space contained, in fact, valuable foresight. He observes that Steiner's result realised more complicated transformations, such as Cremona transformations:

> In his classic [geometrischer Gestalten, Steiner] established and discussed ... the so-called skew projection (Scheife Projektion) and its applications. This projection is based upon two fixed planes, (x) and (x'), and two fixed axes, l and y in space. From every point x in (x) there is, in general, one transversal through l and y which cuts (x') in a point x'. Thus to every point in (x) there corresponds a point in (x'), and conversely. To lines correspond conics, etc. By this construction there is established a general quadratic transformation between two planes, with distinct real and fundamental points and lines in both planes. On page 295, Steiner indicates the quadratic transformation between two spaces, and in a footnote he ... clearly [realises] the possibility of transforma-

tions of a higher order, including Cremona transformations beyond the quadratic.

Further discussion of Steiner and Poncelet's work with quadratic transformations is found in [24]. In this article, the authors discuss the different methods by which Steiner and Poncelet established quadratic transformations and their keen interest in doing so. It is again noted that Steiner appeared to anticipate more general transformations than quadratic ones. Skew projection is not mentioned specifically but perhaps one can infer that skew projection was just one of the means by which Steiner and Poncelet effected a quadratic transformation.

1.3 Skew projection

Previously, we have mentioned skew projection as merely an abstract concept. In this section, we will explain the construction so that the reader will not only understand skew projection in more concrete terms but also in the form relevant to this thesis. In the following, refer to Figure 1.3 for reference and note that the proof of assertions made in this set-up will be given in Section 3.1.

To construct a skew projection, we begin with three mutually *skew* – that is, non-intersecting – lines in three-dimensional space. Let the lines be called l, m, and n. Figure 1.3 is drawn in three-dimensional Euclidean space $\mathbb{R}^3$ but the construction also holds in the projective space $\mathrm{PG}(3, \mathbb{R})$ (projective geometry will be covered in more detail in Section 2.1). From a point on l, call it P, we can always construct a unique transversal, t_P, to all three lines. This transversal will intersect a fixed plane at a point, call it Q_P. Skew projection is the mapping $P \mapsto Q_P$. We are interested in skew projection because as we move P (linearly) along l, the point Q_P traces a conic (conics will be covered in Section 2.2). This construction can be considered as *incidence geometry* because we are only considering points and lines.

The generation of the conic is at first counter-intuitive and moreover, it is not immediately obvious how the conic has been generated. The transformation that is taking place cannot be a collineation (more on these in Section 2.2), because if so, the degree of the curve would be preserved (that is, a collineation cannot map a line to a conic). It would appear that we have generated a conic without intersecting a plane with a cone or by solving an equation (two means by which a conic is usually generated). We have also derived a correspondence between a line and a conic, which shows that the group of the line and the conic is the same. This makes skew projection very interesting.

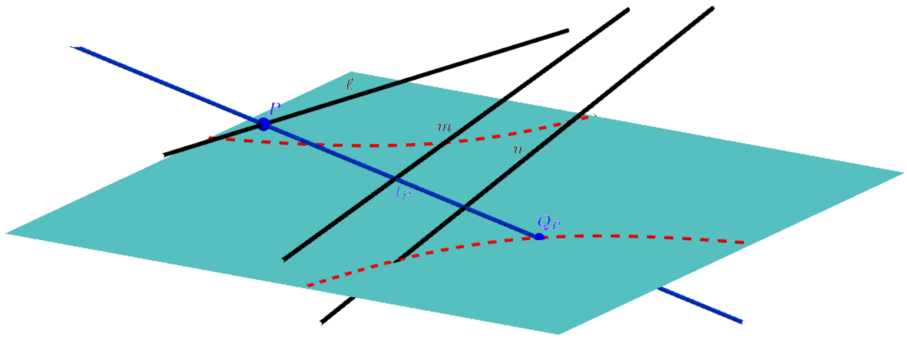

Figure 1.3: A skew projection. Note that the red dashed line indicates the locus of the point Q_P under the skew projection $P \mapsto Q_P$ (that is, the red dashed line shows how the point Q_P moves as we move P). In this case, the conic generated by skew projection is a *hyperbola* (a non-degenerate conic).

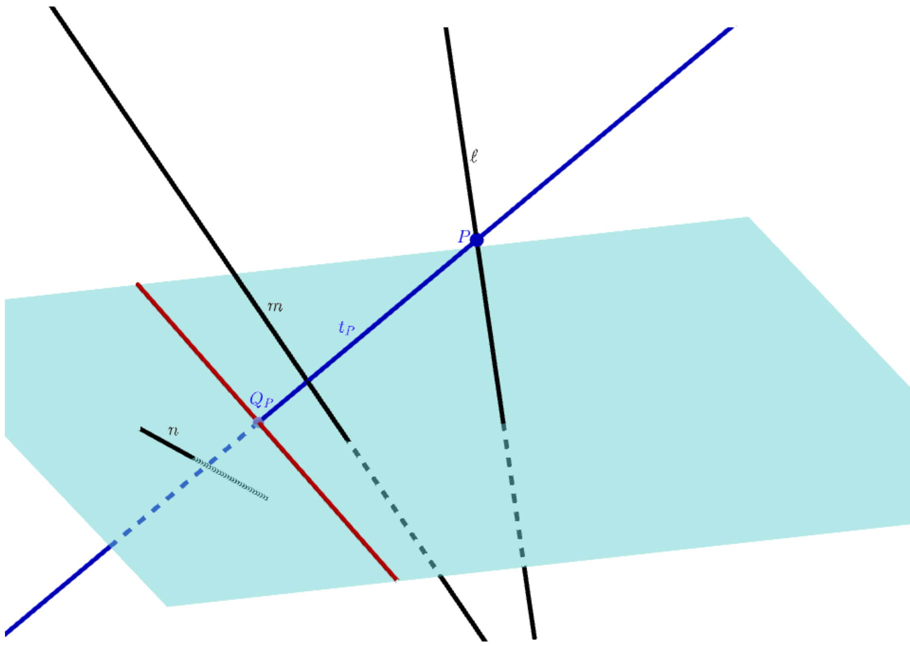

Figure 1.4: We can view this set-up as something of a 'degenerate' skew projection. Here, the lines l, m, n meet the underlying plane $z = 0$ in three collinear points. In this case, the locus of the point Q_P is a line in the underlying plane, so the conic generated is two intersecting lines (a degenerate conic). This may not be immediately obvious from the picture, but we will show this in Theorem 4.3.10.

Chapter 2

Background information

In this chapter, we survey the background theory that will be necessary to proceed to the main results. We provide a brief introduction to projective geometry and then introduce conics, forms, reguli, and quadrics. Finally, we give a brief overview of group actions.

2.1 Projective geometry

Projective geometry grew out of a desire to resolve Euclid's fifth postulate. Euclid established (that which came to be known as) Euclidean geometry with the following five postulates, appearing first in *The Elements* around 300 BC [14]:

1. To draw a straight line from any point to any other;
2. To produce a finite straight line continuously in a straight line;
3. To describe a circle with any centre and distance;
4. That all right angles are equal to each other; and
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

From the beginning, the fifth postulate was not well received. Euclid himself proved the first 28 propositions of *The Elements* without it and through the years many mathematicians provided (what were later realised to be) 'false proofs' of the postulate. The fifth postulate is sometimes stated as Scottish physicist and mathematician John Playfair's formulation (1795), *Through a given point, not on a given straight line, can be drawn only one straight line parallel to the given line* (although this formulation has actually been known since Proclus, a philosopher from the 5th Century BC). Proclus showed that a proof given by Ptolemy to be false and then gave his own false proof. The Italian geometer Girolamo Saccheri gave his false proof in 1697. He assumed the fifth postulate false, derived many results of non-Euclidean geometry, constructed a 'point at infinity', and inadvertently discovered projective geometry [14].

Figure 2.1: Train tracks appearing to meet at the horizon. In projective geometry, we define this perceived point of intersection as a point on the line at infinity. Source: [27]

In Euclidean geometry, parallel lines by definition do not intersect. However in projective geometry, parallel lines do intersect – very far away. More precisely, parallel lines of the Euclidean plane intersect at the 'line at infinity', a line that is 'added' to Euclidean space. A heuristic way of understanding the construction is to recall how, for example, (parallel) train tracks appear to come together at the horizon (an indefinitely long distance away), as can be seen in Figure 2.1.

But why consider projective geometry at all? For one thing, it came as an inevitable result of the work of previous mathematicians to resolve the fifth postulate – failing this, they considered the implications of a geometry without it. Projective geometry offers some advantages to Euclidean geometry, especially when generalising results. When commenting on Desargues's work in projective geometry, Boyer and Merzbach expressed the opinion that '...projective geometry had a tremendous advantage in generality over... metric geometry... for many special cases of a theorem blend into one all-inclusive statement' [7]. Whichever geometry a mathematician chooses to use is up to personal preference and the problem at hand. As Henri Poincaré noted, 'One geometry cannot be more true than another; it can only be more convenient' [13]. For us, projective geometry offers a unified view of conics which we will use to our advantage.

This treatment of projective geometry follows Rey Casse's textbook [9], Chapters 2 and 3. As a preliminary step, we begin by creating the extended Euclidean plane (EEP), an extension of $\mathbb{R}^2$. In the EEP, a line l is the set containing l and all Euclidean lines parallel to l: we call this a 'pencil' of parallel lines. To each pencil we add P_∞, the 'point at infinity of the pencil'. The lines of the pencil intersect at

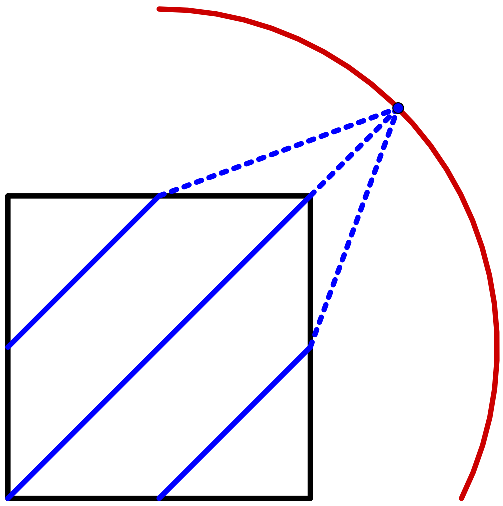

Figure 2.2: A visualisation of the Extended Euclidean Plane. The three blue lines are parallel lines in the Euclidean plane and they intersect at one point on the line at infinity (red curve).

P_∞. We call l with its point at infinity an 'extended line', denoted l^*, and distinct pencils of lines have distinct points at infinity. The set of all points at infinity is the 'line at infinity', denoted l_∞.

We can more rigorously define the EEP as a geometric triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ is the set containing the points of $\mathbb{R}^2$ and all points at infinity, $\mathcal{L}$ is the set containing the extended lines and the line at infinity, and $\mathcal{I}$ is an incidence relation. For the incidence relation $\mathcal{I}$, we stipulate that a point P not at infinity lies on an extended line l^* if and only if P lies on l; a point at infinity P_∞ lies on an extended line l^* if and only if P_∞ is the point at infinity for the pencil l; and all the points at infinity lie on the line at infinity.

We can also generalise the notion of the EEP to higher dimensions, in particular, we can extend $\mathbb{R}^3$. The extended three-dimensional Euclidean space, which we call ES_3, is similar to the EEP, except that we have the additional structure of a 'plane at infinity'. Here 'S_3' alludes to the more general case of an r-dimensional projective space S_r (which we will see later in this section). In ES_3, a plane Π is the set containing Π and all Euclidean planes parallel to Π. We call this a 'pencil' of parallel planes. A line l is the set containing l and all Euclidean lines parallel to l. We call this a 'bundle' of parallel lines. The plane Π with its points and lines at infinity is called an 'extended plane' Π^* and each pencil of parallel planes has a unique line at infinity. The plane at infinity is itself a geometric triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ with $\mathcal{P}$ being the points at infinity, $\mathcal{L}$ being the lines at infinity, and $\mathcal{I}$ the inherited incidence relation. To summarise, points are the points of $\mathbb{R}^3$ and the points of the plane at infinity, planes are pencils of planes with the plane at infinity, lines are bundles of lines with the lines of the plane at infinity, and incidence is inherited.

There are some properties of the subspaces of ES_3 which are worth mentioning. Points, lines, and planes are the proper subspaces of ES_3 and the trivial subspaces are the empty set $\varnothing$ and ES_3. For a subspace S, its dimension is denoted $\dim(S)$.

S	$\dim(S)$
{point}	0
{line}	1
{plane}	2
$\varnothing$	-1
ES_3	3

Table 2.1: The dimensions of the proper and trivial subspaces of ES_3.

The dimension of the proper and trivial subspaces are listed in Table 2.1. For two subspaces S and S', the span of the two subspaces is the intersection of all subspaces of ES_3 containing them and is denoted $\langle S, S'\rangle$. Two subspaces intersect in a subspace and the *Grassmann identity* holds: for every two subspaces S and S' of ES_3, we have

$$\dim\langle S, S'\rangle = \dim(S) + \dim(S') - \dim(S \cap S'). \tag{2.1}$$

More generally, we have the following axioms to define an r-dimensional projective space S_r, for $r \geqslant 2$, which can be found in [9] (Definition 3.7).

1. S_r is a set whose elements are called points;
2. There exists subsets S_h of S_r for every integer h and S_h is a subspace of dimension h;
3. There is a unique subspace S_{-1} of S_r, called the empty set and denoted $\varnothing$. The other trivial subspace is the whole space S_r and all other subspaces of S_r are proper subspaces.
4. Points are the only subspaces with dimension zero;
5. S_r is the unique subspace of dimension r;
6. If S_h and S_k are two subspaces of S_r, then $S_h \subseteq S_k$ if and only if $h \leqslant k$ and $S_h = S_k$ if and only if $h = k$;
7. Given two subspaces S_h and S_k of S_r, their intersection will be contained in S_r;
8. Given two subspaces S_h and S_k of S_r, their span $\langle S_h, S_k\rangle$ is the intersection of all subspaces containing S_h and S_k. If $S_h \cap S_k = S_i$ and $\langle S_h, S_k\rangle = S_c$, then $h + k = i + c$;
9. (Fano's postulate) Every one-dimensional subspace of S_r contains at least 3 points.

Thus, as a result of these axioms, we acquire a simpler formulation of the projective plane Π as a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of subsets of $\mathcal{P}$, called lines, such that: every two points contain a unique line; every two lines contain a unique point; there are at least three non-collinear points in the plane; and there are at least three points on each line. The smallest projective plane is the so-called Fano plane, containing seven points and seven lines (see Figure 2.3).

We have heretofore introduced abstract projective spaces. The work in this thesis is conducted in a projective space over a field, that is an $(n-1)$-dimensional vector space over a field. This is referred to notationally as $\mathrm{PG}(n, \mathbb{F})$, where n is the dimension of the projective space and $\mathbb{F}$ is the underlying field. Most of the work in this thesis is done in $\mathrm{PG}(2, \mathbb{R})$ and $\mathrm{PG}(3, \mathbb{R})$.

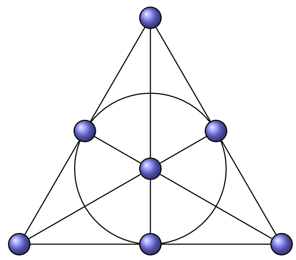

Figure 2.3: The Fano Plane, the smallest projective plane [3].

We can think of $\mathrm{PG}(2,\mathbb{R})$ as two-dimensional Euclidean space extended by the added line at infinity, $z = 0$. The point (x, y, z) in $\mathrm{PG}(2,\mathbb{R})$ is the same point as $p(x, y, z) = (px, py, pz)$, for some $p \in \mathbb{R}\backslash\{0\}$, so scalar multiples of points are in equivalence classes. The point $(0, 0)$ in Euclidean two-dimensional space is identified with the point $(0, 0, 1)$ in $\mathrm{PG}(2,\mathbb{R})$ but the point $(0, 0, 0)$ does not exist in $\mathrm{PG}(2,\mathbb{R})$. More generally, we identify a point (x, y) from $\mathbb{R}^2$ with the point $(x, y, 1)$ in $\mathrm{PG}(2,\mathbb{R})$ and likewise, we identify the projective point (x, y, z), $z \neq 0$, with a Euclidean point by dividing through by z:

$$(x, y, z) \mapsto (X, Y) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

Note that any projective point $(x, y, 0)$ is on the line at infinity and therefore does not have an equivalent Euclidean point. $\mathrm{PG}(3,\mathbb{R})$ is similar except that points now have the form (w, x, y, z).

One final word on projective spaces. There is a remarkable result in projective geometry, that the subspaces of a projective space have a dual configuration. This is known as the Principle of Duality. We state these principles as they relate to $\mathrm{PG}(2,\mathbb{R})$ and $\mathrm{PG}(3,\mathbb{R})$, following Theorems 2.11 and 3.4 in [9].

Theorem 2.1.1 (The Principle of Duality in $\mathrm{PG}(2,\mathbb{R})$)**.** *If $\mathcal{T}$ is a theorem valid in* $\mathrm{PG}(2,\mathbb{R})$ *and $\mathcal{T}'$ is the theorem obtained by interchanging points for lines, collinearity for concurrency, and joins for intersections, with all the necessary grammatical adjustments, then $\mathcal{T}'$ is a valid theorem in* $\mathrm{PG}(2,\mathbb{R})$, *which we call the* dual theorem.

Theorem 2.1.2 (The Principle of Duality in $\mathrm{PG}(3,\mathbb{R})$)**.** *If $\mathcal{T}$ is a theorem valid in* $\mathrm{PG}(3,\mathbb{R})$ *and $\mathcal{T}'$ is the theorem obtained by interchanging points for planes, lines for lines, collinear points for planes in a pencil, and joins for intersections, with all the necessary grammatical adjustments, then $\mathcal{T}'$ is a valid (dual) theorem in* $\mathrm{PG}(3,\mathbb{R})$.

2.2 Conics

Mathematicians have been studying conics since, it is believed, Ancient Greece. The first work on conics is believed to have been written by Menaechmus (ca. 300 BC) [7], although the manuscript has not survived, and his work was continued by Euclid, Archimedes, and most notably Apollonius of Perga, who wrote eight volumes on conics. Around AD 1000 the Persian mathematicians studied conics, including Al-Kuhi and Omar Khayyám. Another 600 years after this, the European mathematicians began to study conics, including Johannes Kepler, Girard Desargues, Blaise Pascal, René Descartes, Pierre Fermat, John Wallis, and Jan de Witt. An influential work on the European studies of conics was Apollonius's eight volume work (we know, for example, that Fermat studied this work). Hence, we know a lot about conics. Yet it is still not immediately obvious how a conic is generated by skew projection and this makes skew projection particularly interesting.

Conic sections usually arise as the intersection of a plane with a cone. There are three non-degenerate conics – ellipses, hyperbolas, and parabolas – and there are three degenerate conics – points, intersecting lines, and double-lines. In projective coordinates, conics satisfy the equation

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0 \tag{2.2}$$

(note that by setting $z = 1$ one retrieves the equation for a conic in Euclidean space). This conic equation is in *homogeneous* form. Let f be a polynomial of degree n in three variables. Then we call $f(x, y, z) = \sum_{i,j=1}^{n} a_{ij}x^i y^j z^{n-i-j}$ homogeneous, meaning that the total power of the exponents in each term of the sum is constant and equal to n. This definition naturally generalises to polynomials of degree n in m variables.

Example 2.2.1. *Consider the polynomial $f(x, y, z) = x^2 + 2xy + 4z^2$. Then f is homogeneous because each term has total power two. Now consider $g(x, y) = x^2 + 2xy + 4$. We see that g is not homogeneous because the last term 4 has total power zero, whereas the other two terms have total power two.*

Let f be a polynomial in $\mathbb{F}^n$. The *projective completion* of f is $\tilde{f} \in \mathrm{PG}(n, \mathbb{F})$, in which the 'additional variable' z is used to balance the polynomial, in the sense that $\tilde{f}$ becomes homogeneous.

Example 2.2.2. *The polynomial $f(x, y) = x^2+y^2$ has projective completion $\tilde{f}(x, y, z) = x^2 + y^2$. There is no change because f is already homogeneous. However, consider the polynomial $g(w, x, y) = w^2 + 2x^2 + 4y + 1$. Then g has projective completion $\tilde{g}(w, x, y, z) = w^2 + 2x^2 + 4yz + z^2$, where $4y \mapsto 4yz$ and $1 \mapsto z^2$ to ensure that the total power of each term is constant.*

Back to conics. Up to a projective transformation, there are five different types of conics in projective space, four degenerate and one non-degenerate (for more information, see [6], Theorem 5.1, and Figure 2.4). They are:

1. $x^2 + y^2 = 0$, a point;
2. $x^2 = 0$, a double-line;
3. $x^2 - y^2 = 0$, two distinct lines;
4. $x^2 + y^2 + z^2 = 0$, the empty set; and
5. $x^2 + y^2 - z^2 = 0$, the circle.

Note then that the circle is the only non-degenerate second degree curve, up to transformation. To the Euclidean mindset, this may at first seem odd. Since projective spaces have more symmetry than Euclidean space, the four non-degenerate

Euclidean conics most people are familiar with (circles, ellipses, parabolas, hyperbolas) are equivalent up to transformation of their defining equations. In projective geometry, we often consider objects up to transformation, specifically, up to *collineation*. A collineation is an injective map between projective spaces which preserves collinearity, that is, the images of collinear points are collinear. As such, a collineation will preserve any property that can be represented by a linear map, for example, a collineation can be used to scale an equation or as a change of basis.

This brings us to the following theorem:

Theorem 2.2.3. *The group of collineations of* $\mathrm{PG}(2,\mathbb{R})$ *is* $\mathrm{PGL}(3,\mathbb{R})$.

Here, $\mathrm{PGL}(3,\mathbb{R})$ is the projective general linear group (for more information, see Section 2.6). We take this theorem without proof, however, we will provide an example.

Example 2.2.4. *Consider the two conics* $\mathcal{C}_1 : x^2+2xy+4z^2=0$ *and* $\mathcal{C}_2 : x^2+y^2-z^2=0$. *Then the matrix* $M \in \mathrm{PGL}(3,\mathbb{R})$ *given by*

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}$$

is a collineation mapping $\mathcal{C}_1$ *to* $\mathcal{C}_2$. *To see this, first recognise that we can represent the conic* $\mathcal{C}_1$ *by the matrix* $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2+2xy+4z^2.$$

Now,

$$MA_1M^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = A_2,$$

which is a representative matrix for $\mathcal{C}_2$:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2+y^2-z^2.$$

Projective space provides us with unique insight into the three different non-degenerate conics (the ellipse, parabola, and hyperbola): we can distinguish each non-degenerate conic by the number of times it intersects the line at infinity.

In $\mathrm{PG}(2,\mathbb{R})$, the ellipse has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$. By substituting $z=0$, we see that for real x, y, they must be both also equal to zero. The point $(0,0,0)$, as previously mentioned, does not exist in $\mathrm{PG}(2,\mathbb{R})$, so an ellipse does not intersect the

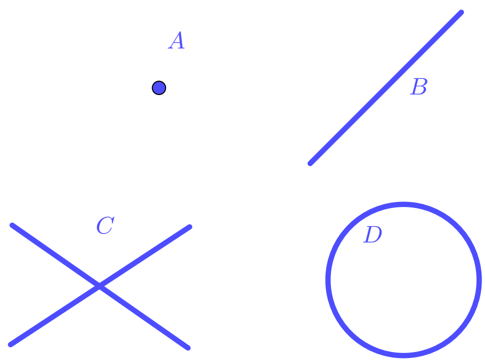

Figure 2.4: The four non-empty projective conics – A: a point, B: a doubled line, C: two distinct lines, and D: the circle.

line at infinity. The parabola $yz = ax^2$ intersects the line at infinity when $x = 0$, so the parabola has a unique intersection with the line at infinity, $(0, 1, 0)$. The hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z^2$, has two points of intersection with the line at infinity, $(1, \pm\frac{b}{a}, 0)$.

The conics generated by skew projection are interesting because they do not appear to be generated by a cone nor have they been forced by an equation. Rather, as we will see in Section 4.3, the conics are arising from a quadric surface. In order to understand this connection, in the following sections we will consider forms, reguli, and quadrics.

2.3 Forms

In this section, we will outline the necessary background on forms. Forms will play a large role in deriving the major results of this thesis as they are inextricably linked to quadric surfaces (see Section 2.5). Two important theorems in this thesis, Sylvester's Law of Inertia (4.1.2) and Witt's Theorem (4.3.3), are theorems on forms.

A *form* is a mapping from a vector space to a field. A bilinear form B over a vector space V over a field $\mathbb{F}$ is a map $B : V \times V \to \mathbb{F}$ that is linear in each coordinate, that is, such that for every λ, $\mu \in \mathbb{F}$,

1. $B(\lambda x + \mu y, z) = \lambda B(x, z) + \mu B(y, z)$ and
2. $B(x, \lambda y + \mu z) = \lambda B(x, y) + \mu B(x, z)$.

Furthermore, a bilinear form is called *symmetric* if $B(x, y) = B(y, x)$.

Example 2.3.1. *Consider the bilinear form B defined by $B((x_1, y_1, z_1), (x_2, y_2, z_2)) = x_1z_2 + z_1x_2 - 2y_1y_2$. Then B is indeed linear in each coordinate:*

$$\begin{aligned}
&B(\lambda(x_1, y_1, z_1) + \mu(x_2, y_2, z_2), (x_3, y_3, z_3))\\
=&B((\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2, \lambda z_1 + \mu z_2), (x_3, y_3, z_3))\\
=&(\lambda x_1 + \mu x_2)z_3 + (\lambda z_1 + \mu z_2)x_3 - 2(\lambda y_1 + \mu y_2)y_3\\
=&\lambda(x_1z_3 + z_1x_3 - 2y_1y_3) + \mu(x_2z_3 + z_2x_3 - 2y_2y_3)
\end{aligned}$$

$$=\lambda B((x_1,y_1,z_1),(x_3,y_3,z_3))+\mu B((x_2,y_2,z_2),(x_3,y_3,z_3))$$

Linearity in the second coordinate is similar. Moreover, B is symmetric, as $B((x_1,y_1,z_1),(x_2,y_2,z_2)) = x_1z_2+z_1x_2-2y_1y_2$ and $B((x_2,y_2,z_2),(x_1,y_1,z_1)) = x_2z_1+z_2x_1-2y_1y_2 = x_1z_2+z_1x_2-2y_1y_2$.

A *quadratic form* $\mathcal{Q}$ is a map $\mathcal{Q}: V \to \mathbb{F}$ such that

1. $\mathcal{Q}(\lambda v) = \lambda^2\mathcal{Q}(v)$, $\lambda \in \mathbb{F}$ and
2. There is a unique associated symmetric bilinear form B such that $B(u,v) = \mathcal{Q}(u+v)-\mathcal{Q}(u)-\mathcal{Q}(v)$.

For a quadratic form over fields not of characteristic two, we have

$$\begin{aligned}B(u,u) = \mathcal{Q}(u+u)-\mathcal{Q}(u)-\mathcal{Q}(u)\\ =&\mathcal{Q}(2u)-2\mathcal{Q}(u)\\ =&4\mathcal{Q}(u)-2\mathcal{Q}(u)\\ =&2\mathcal{Q}(u).\end{aligned}$$

This does not hold over fields of characteristic two, however, as we see in the following example.

Example 2.3.2. *Consider the symmetric bilinear form B defined by $B((x_1,y_1,z_1),(x_2,y_2,z_2)) = x_1x_2$. Suppose we take B over a field of characteristic two. In these fields, $2=0$ and $-1=1$. Consider the vector $v_1=(1,0,0)$. Then $B((1,0,0),(1,0,0)) = (1)(1) = 1$, however, $2\mathcal{Q}((1,0,0)) = 0$.*

Example 2.3.3. *Consider the quadratic form defined by $\mathcal{Q}_1(w,x,y,z) = wz+xy$. Indeed, it easy to verify that $\mathcal{Q}_1(\lambda(w,x,y,z)) = \lambda^2\mathcal{Q}_1(w,x,y,z)$:*

$$\begin{aligned}&\mathcal{Q}_1(\lambda(w,x,y,z))\\ =&\mathcal{Q}_1(\lambda w,\lambda x,\lambda y,\lambda z)\\ =&\lambda w\lambda z+\lambda x\lambda y\\ =&\lambda^2wz+\lambda^2xy\\ =&\lambda^2(wz+xy)\\ =&\lambda^2\mathcal{Q}_1(w,x,y,z).\end{aligned}$$

It follows from computing $\mathcal{Q}_1((w_1,x_1,y_1,z_1)+(w_2,x_2,y_2,z_2))-\mathcal{Q}_1(w_1,x_1,y_1,z_1)-\mathcal{Q}_1(w_2,x_2,y_2,z_2)$ that $\mathcal{Q}_1$ has associated bilinear form defined by $B_1((w_1,x_1,y_1,z_1),(w_2,x_2,y_2,z_2)) = w_1z_2+z_1w_2+x_1y_2+y_1x_2$. Linearity in each coordinate is easy to verify, so B_1 is indeed a bilinear form. Moreover, B_1 is a symmetric bilinear form, since

$$\begin{aligned}&B_1((w_1,x_1,y_1,z_1),(w_2,x_2,y_2,z_2))\\ =&w_1z_2+z_1w_2+x_1y_2+y_1x_2\\ =&z_2w_1+w_2z_1+y_2x_1+x_2y_1\\ =&w_2z_1+z_2w_1+x_2y_1+y_2x_1\\ =&B_1((w_2,x_2,y_2,z_2),(w_1,x_1,y_1,z_1)).\end{aligned}$$

Example 2.3.4. *Another example of a quadratic form in* $\mathbb{R}^3$ *is* $\mathcal{Q}_2$ *defined by* $\mathcal{Q}_2(x,y,z) = xz - y^2$. *Once again it is easy to verify that*

$$
\begin{aligned}
&\mathcal{Q}_2(\lambda(x,y,z))\\
=&\mathcal{Q}_2(\lambda x, \lambda y, \lambda z)\\
=&\lambda x \lambda z - (\lambda y)^2\\
=&\lambda^2(xz - y^2)\\
=&\lambda^2 \mathcal{Q}_2(x,y,z).
\end{aligned}
$$

We find the associated bilinear form by computing $\mathcal{Q}_2((x_1,y_1,z_1)+(x_2,y_2,z_2)) - \mathcal{Q}_2(x_1,y_1,z_1) - \mathcal{Q}_2(x_2,y_2,z_2)$ *and we see that the associated bilinear form for* $\mathcal{Q}_2$ *is the bilinear form defined by* $B_2((x_1,y_1,z_1),(x_2,y_2,z_2)) = x_1z_2 + z_1x_2 - 2y_1y_2$, *which we recognise as the bilinear form introduced in Example 2.3.1. We have previously shown in Example 2.3.1 that this form is linear in each coordinate, hence bilinear, and symmetric.*

We can also represent forms by a matrix (the matrix of a bilinear form is explored in more detail in [21], Chapter 11). Suppose we have a vector space V with a symmetric bilinear form B. Let $\mathcal{B} = (v_1, ..., v_n)$ be an ordered basis for V. Then the bilinear form B is completely determined by its *associated matrix* $M_{\mathcal{B}}$, defined by

$$M_{\mathcal{B}} = (a_{ij})^n_{\{i,j\}=1} = (B(v_i, v_j))^n_{\{i,j\}=1}.$$

Let $(v_1, ..., v_n)$ be a basis for a vector space V endowed with a bilinear form B. Conversely, we can recover the form B from the matrix $M_{\mathcal{B}}$ by computing

$$B(x,y) = xM_{\mathcal{B}}y^\top,$$

for some x, y in V. The associated matrix of a quadratic form is the associated matrix of the bilinear form associated with the quadratic form.

Example 2.3.5. *The quadratic form* $w^2 + x^2 - y^2 - z^2$ *has associated matrix* $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ *with respect to the standard basis* $\{e_1, ..., e_4\}$ *for* $\mathbb{R}^4$.

Example 2.3.6. *Recall the quadratic form* $\mathcal{Q}_1$ *from Example 2.3.3. Let* $\{e_1, ..., e_4\}$ *be the standard basis in* PG$(3,\mathbb{R})$, *where each* e_i *is a* 1×4 *row vector with 1 in the ith column and zeros elsewhere. Then the associated matrix for* $\mathcal{Q}_1$ *with respect to the standard basis is*

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

The set of points P *such that* $P\mathcal{Q}_1P^\perp = 0$ *is a* hyperbolic quadric *in* PG$(3,\mathbb{R})$ *(hyperbolic quadrics will be discussed in more detail in Section 2.5).*

Example 2.3.7. *Recall the quadratic form $\mathcal{Q}_2$ and its associated bilinear form B_2 from Example 2.3.4. $\mathcal{Q}_2$ has associated matrix*

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Recall that a matrix is *symmetric* if it is equal to its transpose. Thus we have the following:

Lemma 2.3.8. *A bilinear form is symmetric if and only if its associated matrix is a symmetric matrix.*

Proof. Let B be a bilinear form over a vector space V and let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V. The associated matrix $M_{\mathcal{B}}$ of B with entries $m_{ij} = B(v_i, v_j)$ will be symmetric if and only if $B(v_i, v_j) = B(v_j, v_i)$ for every i, j. This holds if and only if B is a symmetric bilinear form. Thus the form B is symmetric if and only if its associated matrix $M_{\mathcal{B}}$ is symmetric. Q.E.D.

We call a bilinear form *non-degenerate* if the determinant of its associated matrix is non-zero. We call two forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$, having associated matrices A and B, respectively, *congruent* if there exists a matrix P such that $B = PAP^{\top}$, and the matrices A, B are said to be in the same *congruency class*. Likewise, we call two forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$, having associated matrices A and B, *similar* if there exists a matrix P such that $B = PAP^{-1}$, and the matrices A, B are said to be in the same *similarity class*.

In this work, bilinear forms will be of the kind $B : \mathrm{PG}(3, \mathbb{R}) \times \mathrm{PG}(3, \mathbb{R}) \to \mathbb{R}$ and quadratic forms will be of the kind $\mathcal{Q} : \mathrm{PG}(3, \mathbb{R}) \to \mathbb{R}$ (note that perturbing the domains in this manner does not affect the form).

Let B be a bilinear form over a vector space V. Two vectors v_i, $v_j \in V$ are said to be *orthogonal* if $B(v_i, v_j) = 0$.

Example 2.3.9. *Recall the bilinear form from Example 2.3.3,*
$B_1((w_1, x_1, y_1, z_1), (w_2, x_2, y_2, z_2)) = w_1z_2 + w_2z_1 + x_1y_2 + x_2y_1$. *Then the points* $(1, 0, 0, 1)$ *and* $(-1, 0, 0, 1)$ *are orthogonal with respect to* B_1*, since*
$B_1((1, 0, 0, 1), (-1, 0, 0, 1)) = (1)(1) + (-1)(1) + (0)(0) + (0)(0) = 0.$

2.4 Reguli

In this section we define a *regulus* and give some examples. Before formally defining the regulus, a special note should be made about *transversals*. A transversal is a line that intersects each line in a system of lines once. From a fixed point on one line in a system of two lines, there are infinitely many constructions of a transversal to the system. However, when there are three skew lines in the system, there is only one choice of transversal to the system from a fixed point (see Figures 2.5 and 2.6). The uniqueness of such a transversal will be covered in Section 3.1.

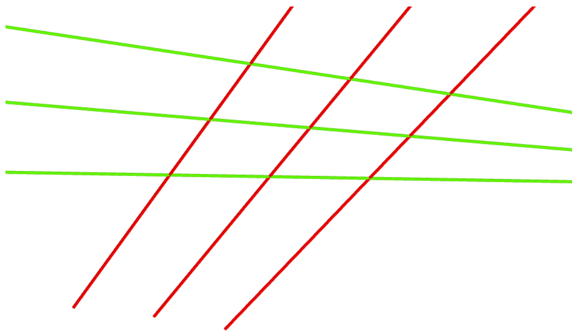

Figure 2.7: If the green lines are taken to be the set of skew lines, we have here a regulus (red lines) and its opposite regulus (green lines) in $\mathbb{R}^3$.

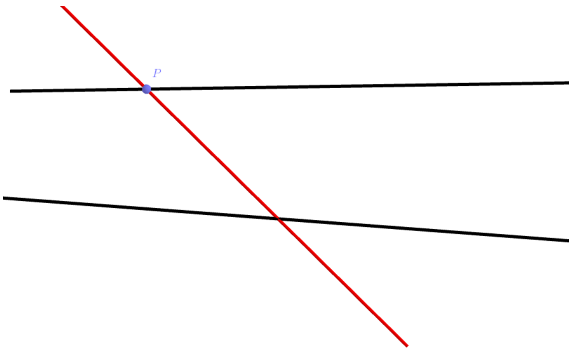

Figure 2.5: A transversal to two lines through a fixed point P.

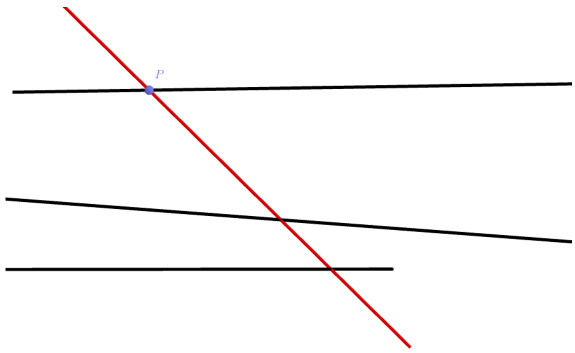

Figure 2.6: The unique transversal to three lines through a fixed point P.

Briefly, if l, m, n are three mutually disjoint (skew) lines, then their regulus is the set containing their transversals, that is $\mathcal{R}(l, m, n) = \{\text{transversals to } l,\ m,\ n\}$.

In [5], Section 2.4, reguli are defined with respect to a three-dimensional projective space as being non-empty sets $\mathcal{R}$ of skew lines such that there exist transversals to $\mathcal{R}$ through each point of each line of $\mathcal{R}$ and that, vice versa, through every point of each transversal of $\mathcal{R}$ there is a line of $\mathcal{R}$. It is clear, then, that the regulus of the regulus, the *opposite regulus* $\mathcal{R}'$, is itself a regulus. But note that in an arbitrary geometry, the opposite regulus is not always the original set of skew lines, indeed, it is a proper superset of the original skew lines. Equality of the opposite regulus and the three skew lines is a special property of the hyperbolic quadric (see Section 2.5).

Reguli arise in a beautiful correspondence between geometry axioms and algebra. A theorem in [5] (2.4.3 in their text) which outlines properties of reguli is proved using the Dandelin-Gallucci Theorem, recorded in their text as the 16 point theorem (2.4.2). The Dandelin-Gallucci theorem requires the additional structure that the regulus is constructed in a 3-dimensional projective space over a *division ring*. A division ring is a set F with operations $+, \cdot$, such that $(F, +)$ is a commutative group, $(F\backslash\{0\}, \cdot)$ is a not necessarily commutative group, and left and right distributivity holds ([5]). Note, of course, that $\mathbb{R}$ is a division ring because a field is a commutative division ring. In [5], the Dandelin-Gallucci Theorem appears thus:

Theorem 2.4.1 (The Dandelin-Gallucci 16 point theorem)**.** *Let $\mathbb{P}$ be a 3-dimensional projective space over the division ring F. Let $\{g_1, g_2, g_3\}$ and $\{h_1, h_2, h_3\}$ be sets of skew lines with the property that each line g_i meets each line h_j. Then the following is true: F is commutative (hence a field) if and only if each transversal $g \notin \{g_1, g_2, g_3\}$ of $\{h_1, h_2, h_3\}$ intersects each transversal $h \notin \{h_1, h_2, h_3\}$ of $\{g_1, g_2, g_3\}$.*

Sketch of proof, following [5]. Firstly, we define the points $\langle v_1\rangle = g_1 \cap h_1$, $\langle v_2\rangle = g_1 \cap h_2$, $\langle v_3\rangle = g_2 \cap h_1$, and $\langle v_4\rangle = g_2 \cap h_2$ (see Figure 2.8). It follows that $g_3 \cap h_1 = \langle av_1 + bv_3\rangle$ for some $a, b \in \mathbb{F}\backslash\{0\}$. Without loss of generality, we can assume that $b = 1$. Then, by replacing v_1 by $v_1' = av_1$, we have $g_3 \cap h_1 = \langle v_1' + v_3\rangle$. With suitable scaling, we can take $v_1' = v_1$, so $g_3 \cap h_1 = \langle v_1 + v_3\rangle$.

Similarly, we can assume that $g_1 \cap h_3 = \langle v_1 + v_2\rangle$ and $g_3 \cap h_3 = \langle v_3 + v_4\rangle$, so $g_3 \cap h_2 = \langle v_2 + av_4\rangle$ for some $a \in \mathbb{F}\backslash\{0\}$.

By assumption, each of the skew lines $\{g_1, g_2, g_3\}$ intersects $\{h_1, h_2, h_3\}$, so g_3, h_3 have the unique intersection

$$g_3 \cap h_3 = \langle a_1(v_1 + v_3) + a_2(v_2 + av_4)\rangle = \langle b_1(v_1 + v_2) + b_2(v_3 + v_3)\rangle.$$

Since v_1, v_2, v_3, v_4 are linearly independent, we have $a_1 = a_2 = b_1 = b_2$ and $a = 1$, so we can assume $g_3 \cap h_3 = \langle v_1 + v_2 + v_3 + v_4\rangle$ and $g_3 \cap h_2 = \langle v_2 + v_4\rangle$.

Define the lines $g = \langle v_i + av_j\rangle$, $h = \langle v_l + bv_k\rangle$ for some a, $b \in \mathbb{F}\backslash\{0\}$. The theorem then follows by showing that $g \cap h$ have non-empty intersection if and only if $ab = ba$. Further detail is beyond the scope of this thesis, but briefly, it is a fact that there is at most one line through $g \cap h_1$ intersecting both h_2 and h_3 and this line must be g, so we find that $g = \langle v_1 + av_3, v_2 + av_4\rangle$, and by similar reasoning we find that $h = \langle v_1 + bv_2, v_3 + bv_4\rangle$. Then computing the intersection of g and h will show that they have a common point if and only if $ab = ba$, that is, the division ring is commutative. Q.E.D.

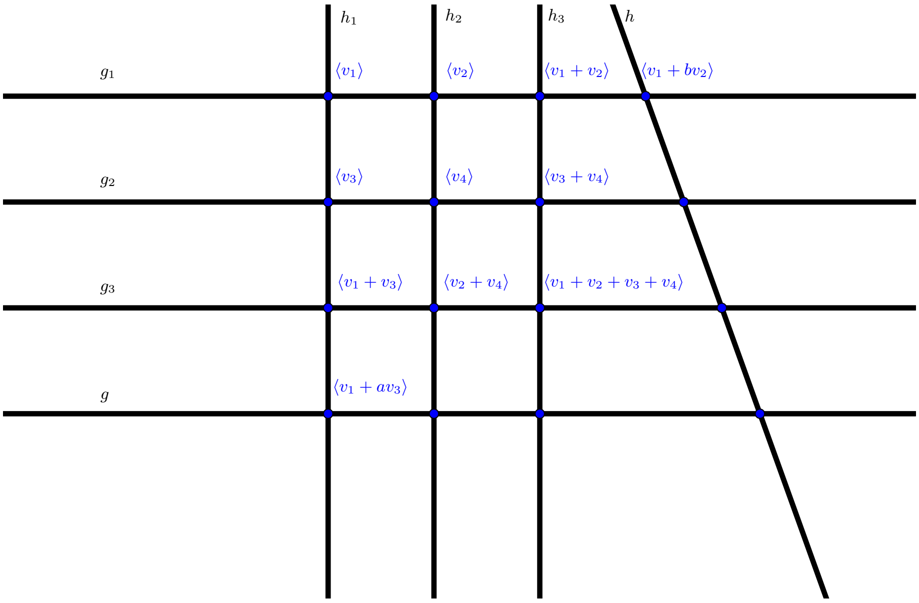

Figure 2.8: A reproduction of a figure from [5] (Figure 2.3) showing the set-up of the proof of Theorem 2.4.1.

We now present the theorem on reguli from [5] (Theorem 2.4.3). It is clear that the second and third properties of reguli in this theorem follow immediately from Theorem 2.4.1, whilst the first property is a simple argument based on the definitions of reguli and transversals.

Theorem 2.4.2. *Let* $\mathrm{PG}(3, \mathbb{D})$ *be a 3-dimensional projective space over a division ring* $\mathbb{D}$*. Let* l*,* m*,* n *be three skew lines of* $\mathrm{PG}(3, \mathbb{D})$*. Then the following assertions are true:*

1. *There is at most one regulus containing* l*,* m*, and* n*.*
2. *If* $\mathbb{D}$ *is noncommutative then there is no regulus in* $\mathrm{PG}(3, \mathbb{D})$*.*
3. *If* $\mathbb{D}$ *is commutative, then there is exactly one regulus through* l*,* m*, and* n*.*

In our case of a projective space over $\mathbb{R}$, this theorem tells us that each triple of skew lines in $\mathbb{R}^3$ will determine a unique regulus. The uniqueness of the transversal through the three skew lines l, m, n comes as a corollary of the uniqueness of the regulus determined by l, m, n.

Another approach to constructing a regulus is taken in [18] (Section 83.1). Here the regulus is constructed using a *projectivity* or *projective correspondence*. Projective correspondences will not be covered in this thesis in detail, but briefly, subsets in a projective space are said to have a projective correspondence between them if there exists a projective transformation between them. If a projectivity is established between the planes of pencils of two skew lines in $\mathrm{PG}(3, \mathbb{R})$, then the regulus is thus defined as the set of lines which are the intersections of corresponding planes in the projectivity (see Figure 2.9).

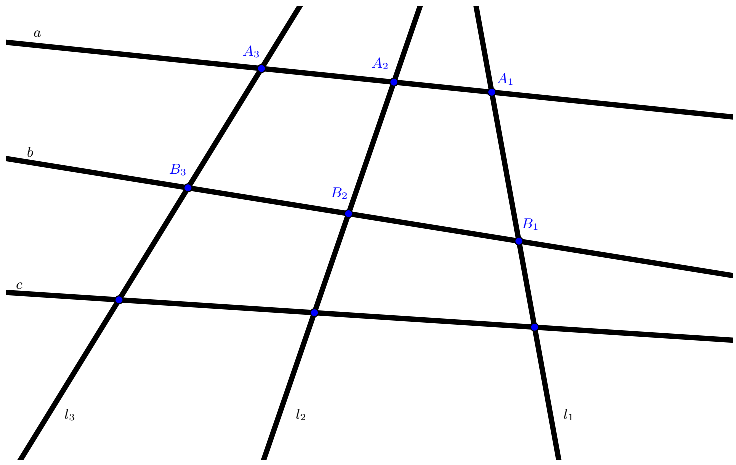

Figure 2.9: A reproduction of a figure in [18] (Figure 83.2) which presents the construction of a regulus through projectivity. Here a and b are skew lines and the points A_1, A_2, A_3 on a correspond to the points B_1, B_2, B_3 on b via a projectivity. The lines l_1, l_2, l_3 are skew and the transversal c to these lines is skew to a and b. Continuing in this manner, we form a regulus.

2.5 Quadric surfaces

A *quadric surface* or *quadric* is the *kernel* of a quadratic form (the set of points mapped to zero by the quadratic form). If we have a quadratic form $\mathcal{Q}$ over a projective space $\mathrm{PG}(3, \mathbb{F})$, then we call the set of points $(w, x, y, z) \in \mathrm{PG}(3, \mathbb{F})$ such that $\mathcal{Q}(w, x, y, z) = 0$ a *projective quadric*. If $\mathcal{Q}$ is instead over Euclidean space $\mathbb{R}^n$, then we call the set of points $(x, y, z) \in \mathbb{R}^n$ such that $\mathcal{Q}(x, y, z) = 0$ an *affine quadric*. In this section, we will begin by working through some examples of affine and projective quadrics and proceed to explain various presentations of quadrics as they appear in the literature.

Example 2.5.1. *An example of a projective quadric is the elliptic quadric, the set of points which satisfies the equation* $w^2 + x^2 + y^2 - z^2 = 0$*. A point on this surface is* $(1, 0, 0, 1)$*. We will see in Theorem 4.1.1 that the elliptic quadric is one of only two non-empty non-degenerate quadrics in* $\mathrm{PG}(3, \mathbb{R})$*, up to projective equivalence.*

Example 2.5.2. *The affine equivalent of Example 1 is the ellipsoid, the set of points satisfying the equation* $-x^2 - y^2 - z^2 + 1 = 0$*. A point on this surface is* $(1, 0, 0)$*. Note that this point corresponds to the projective point* $(1, 0, 0, 1)$*.*

A neat construction of the hyperbolic quadric can be found in [5] (Theorem 2.4.4). Take $S_1 = \langle (1, 0, 0, 0),\ (0, 1, 0, 0) \rangle$, $S_2 = \langle (0, 0, 1, 0),\ (0, 0, 0, 1) \rangle$, $S_3 = \langle (1, 0, 1, 0),\ (0, 1, 0, 1) \rangle$ to be the three *fundamental skew lines* of $\mathrm{PG}(3, \mathbb{R})$. As was discussed in Section 2.4, three skew lines uniquely determine a regulus. Let the regulus determined by the three fundamental skew lines be $\mathcal{R}$. Then the set of points on $\mathcal{R}$ satisfy

$$\mathcal{Q} = \{(w, x, y, z) \in \mathrm{PG}(3, \mathbb{R}) : xw = yz\}.$$

Each point of $\mathcal{Q}$ satisfies the quadratic equation $xw = yz$ and we call $\mathcal{Q}$ the hyperbolic quadric of $\mathrm{PG}(3, \mathbb{R})$. The proof supplied in [5] is relatively simple, however, it relies on properties of reguli that were proved using the more powerful Dandelin-Gallucci theorem (Theorem 2.4.1). It is clear from this construction that the hyperbolic quadric contains lines. When we classify the non-empty non-degenerate projective quadrics in Theorem 4.1.1, we will see that the two distinct non-degenerate non-empty quadrics up to projective equivalence differ in that one contains lines (the hyperbolic quadric) and one does not (the elliptic quadric).

The quadric in [18] (Section 81.1) is constructed by a set of skew lines and their transversals (at first not stating these to be reguli). Transversals l', m', n' are constructed to three skew lines l, m, n through three distinct points on n and then transversals to the transversals l', m', n' are constructed. The intersection of these transversals are points lying on the surface $xw - yz$, a quadric surface. The surface contains two systems of generators (lines lying in the surface), one system containing l, m, n, the other containing l', m', n', and any line in one system of transversals intersects any line in the other system of transversals – that is, the generators are reguli.

2.6 Group actions

Many of the proofs in this thesis rely on group actions. It will be helpful to give brief definitions of the most important concepts, however more detailed discussions can be found in various introductory textbooks, for example [2] (Chapters 2, 5, and 6).

Let G be a group and let X be a set. Let $g \in G$, $x \in X$ and let 1_G denote the identity element of G. Then G *acts* on X if there is a map $G \times X \to X$ such that:

1. $x^{1_G} = x$ for all $x \in X$, and
2. $x^{gg'} = (x^g)^{g'}$ for all $g, g' \in G$ and for all $x \in X$.

We call the map $x \mapsto x^g$ the *action* of G on X. Note that we denote group actions exponentially, whereas some introductory texts, such as [2], denote them multiplicatively.

Example 2.6.1. *Let G be a group and let X be a set. Then G acts on X by right multiplication, defined by $x^g = xg$, for every $g \in G$ and for every $x \in X$. To see that this is indeed an action, we check the two conditions:*

1. *We have $x^{1_G} = x1 = x$, and*
2. *$(x^g)^h = (xg)^h = xgh$, while $x^{gh} = xgh$.*

Both conditions are satisfied, so right multiplication is a group action for every group G on every set X.

We can consider how a group action 'moves' the elements in the set it acts on. As such, let X be a set. The *orbit* of an element $x \in X$ with respect to a group G is the set of elements $x' \in X$ such that $x' = x^g$ for some $g \in G$. We often denote this set x^G, that is $x^G = \{x' \in X : x' = x^g \text{ for } g \in G\}$. Informally, this is the set of all the elements to which an element x in a set X is mapped by the group G under its action on X. We call the action of a group G on a set X *transitive* if X has only one orbit under G. Another way of stating this is that for any x, $y \in X$, there exists some $g \in G$ such that $x^g = y$. It follows, then, that a transitive action has only one orbit.

Example 2.6.2. *We now give a more concrete example of a group action. Let $G = S_2$ be the permutation group of order two and let $X = \{1, 2\}$. Let's examine the action of G on X by permutation. G consists of the two permutations: $()$, the identity permutation that fixes 1, 2, and (12), the permutation taking 1 to 2 and 2 to 1.*

Consider how G acts on the element 1. Under the permutation $()$, we see that $1^{()} = 1() = 1$ and under the permutation (12), we see that $1^{(12)} = 1(12) = 2$. So the orbit of 1 under the action of G by permutation is $\{1, 2\}$, that is, $1^G = \{1, 2\}$. Similarly, it is clear that $2^G = \{1, 2\}$, so $1^G = 2^G$ and the action has only one orbit. It is clear then that the action of G by permutation is transitive: the permutation (12) takes 1 to 2 and 2 to 1, so for every two elements of X, there is an element of G that maps these elements to each other.

Some elements of G will fix certain elements of X. This is particularly interesting because many elements other than the identity could fix elements of the set. The *stabiliser* of an element $x \in X$ under the action of a group G is the set of all elements in the group which fix x. We often denote this set G_x, that is, $G_x = \{g \in G : x^g = x\}$.

Example 2.6.3. *Recall Example 2.6.2. It is clear from our previous computations that the element 1 is fixed only by the identity permutation, so $G_1 = \{()\}$, and it is also clear that the element 2 is fixed only by the identity permutation, so $G_2 = \{()\}$. So only the identity permutation fixes each element of X.*

Example 2.6.4. *Let G be a group, X be a set, and define the* trivial action *by $x^g = x$ for every $x \in X$, $g \in G$. Clearly, $x^{1_G} = x$ and $x^{g^h} = x^h = x = x^{gh}$ for every g, $h \in G$, so the trivial action does indeed define an action of G on X. Since every element of G fixes every element of X, we have for every $x \in X$ that $G_x = G$. Quite the opposite of Example 2.6.3.*

There are two matrix groups which will be of particular importance to this thesis. These are the general linear group and the projective general linear group. We will give a definition and examples of both of these.

The *general linear group*, denoted $\mathrm{GL}(n, \mathbb{F})$, is the group of $n \times n$ invertible matrices over a field $\mathbb{F}$.

Example 2.6.5. *The upper triangular matrices $U = \begin{bmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & * \end{bmatrix}$ over $\mathbb{R}$ are invertible $n \times n$ matrices, so the upper triangular matrices are elements of $\mathrm{GL}(n, \mathbb{R})$. In fact, the upper triangular matrices are a* subgroup *of $\mathrm{GL}(n, \mathbb{R})$ because the* *product of two upper triangular matrices is an upper triangular matrix and the identity matrix is an upper triangular matrix.*

The *projective general linear group*, denoted $\mathrm{PGL}(n,\mathbb{F})$, is the group of $n \times n$ invertible matrices over a field $\mathbb{F}$ such that two matrices are considered equal if they are scalar multiples of each other. As such, $PGL(n,\mathbb{R})$ is the quotient of $GL(n,\mathbb{R})$ by scalar matrices λI, for some $\lambda \in \mathbb{R}\backslash\{0\}$.

Example 2.6.6. *The identity matrix* $I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & \dots & & 1 \end{bmatrix}$ *over* $\mathbb{R}$ *is an invertible* $n \times n$ *matrix, so it is an element of both* $\mathrm{GL}(n,\mathbb{R})$ *and* $\mathrm{PGL}(n,\mathbb{R})$*. However, in* $\mathrm{PGL}(n,\mathbb{R})$*, any scalar multiple of* I *is considered equal to* I*, that is,* $I = \lambda I$ *for any non-zero* $\lambda \in \mathbb{R}$*.*

Chapter 3

Skew projection

In this chapter, we provide a full proof of the earlier assertions that skew projection generates a conic via a quadric. We will prove the existence of the unique transversal through three skew lines and the existence of a hyperbolic quadric generated by three skew lines and their regulus. We will then classify the projective quadrics up to projective equivalence, via Sylvester's Law of Inertia, and finally we will determine the type of conic generated by skew projection via Witt's Theorem.

Firstly, it could be helpful to the reader to develop an overview of the problem. Skew projection consists of three skew (non-intersecting and, in the affine case, non-parallel) lines and their unique transversal through a given point on one of the lines. Moreover, three skew lines generate a unique regulus, the set of transversals to three skew lines. In this case, the opposite regulus, transversals to the regulus, will consist of the given three skew lines and additional transversals to the regulus. The regulus and opposite regulus which we have now constructed generate a hyperbolic quadric. The lines of the hyperbolic quadric fall into two equivalence classes, determined by the two reguli. Planes intersect the quadric in a conic and this is the conic we generate through skew projection.

3.1 A more rigorous proof of the set up

Recall skew projection as outlined in Secton 1.3, that is, we have three mutually skew lines l, m, n and from a point P on l we construct the unique transversal t_P to all three lines. The transversal t_P will intersect a fixed plane π at a point, call it Q_P. Skew projection is the mapping $P \mapsto Q_P$.

Let P be a point on l. The lines l, m are skew, so they do not intersect in any point, hence the point P cannot be on m. Using the Grassmann identity (2.1), we argue that P and m span a plane π_P:

$\dim(P \cap m) = -1$ so,

$$\dim(\langle P, m\rangle) = \dim(P) + \dim(m) - \dim(P \cap m)$$

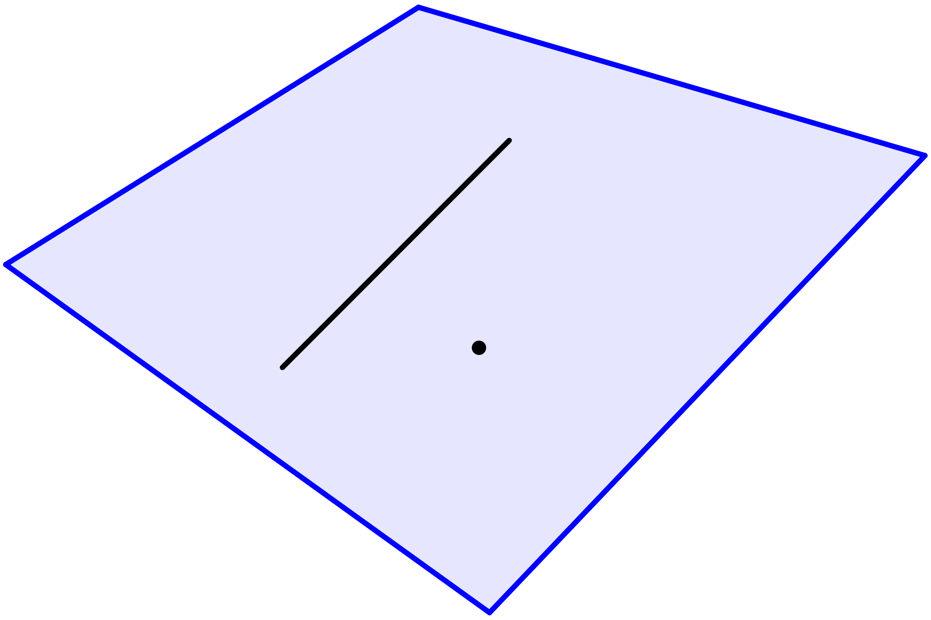

Figure 3.1: A point and a line with empty intersection in projective space span a plane.

$$=0 + 1 - (-1)$$
$$=2,$$

hence P and m span a plane π_P because the subspaces of $\mathrm{PG}(3, \mathbb{R})$ with dimension 2 are planes (see Figure 3.1).

We want π_P to meet the line n in a point. Firstly, note that if n is contained in π_P, then m and n must meet because π_P is the span of P and m – that is, the intersection of all subspaces of $\mathrm{PG}(3, \mathbb{R})$ containing both P and m. But this is a contradiction, since m and n are defined to be skew. So n is not contained in π_P and as a result either n and π_P meet in a point or they do not meet at all. Thus, we show the following lemma:

Lemma 3.1.1. *A line in* $\mathrm{PG}(3, \mathbb{R})$ *is either contained in a plane or it intersects the plane in one point.*

Proof. Let l be a line in $\mathrm{PG}(3, \mathbb{R})$ and let π be a plane. There are three cases: l is disjoint from π, l is contained in π, or l intersects π in a point. If l is disjoint from π, then $l \cap \pi = \varnothing$, so by the Grassmann identity (2.1),

$$\begin{aligned}\dim(\langle l, \pi\rangle) &= \dim(l) + \dim(\pi) - \dim(l \cap \pi)\\ &= 2 + 1 - (-1)\\ &= 4,\end{aligned}$$

and we cannot have a subspace of dimension greater than three. If l is contained in π, then $l \cap \pi = \pi$, so

$$\dim(\langle l, \pi\rangle) = \dim(l) + \dim(\pi) - \dim(l \cap \pi)$$

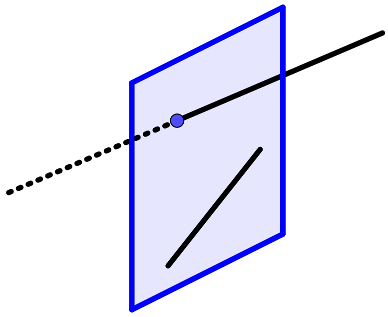

Figure 3.2: A line in projective space either intersects a plane in one point or lies in the plane.

$$
\begin{aligned}
&= 2 + 1 - 2 \\
&= 1,
\end{aligned}
$$

which is valid. Finally, if l intersects π in a point, then $l \cap \pi = Q$ for some point Q, so

$$
\begin{aligned}
\dim(\langle l, \pi \rangle) &= \dim(l) + \dim(\pi) - \dim(l \cap \pi) \\
&= 2 + 1 - 0 \\
&= 3,
\end{aligned}
$$

which is valid. Hence, any line in $\mathrm{PG}(3, \mathbb{R})$ is either contained in a plane or it intersects the plane in one point (see Figure 3.2). Q.E.D.

Hence by Lemma 3.1.1, π_P and n meet in a point, say N_P. Since l is skew to n, the points P and N_P are distinct and span a line. This is why we wanted π_P to meet the line n in a point: we can now construct a line, call it t_P, and we claim that $t_P = \langle P, N_P \rangle$ is a transversal to l, m, n. To see this, we need to check the following three claims:

Claim 1. *The transversal t_P is contained in π_P.*

Proof. The point P is contained in π_P since π_P is the span of P and the line m. The point N_P is contained in π_P because N_P is the intersection of π_P with the line n. Since both P and N_P are contained in π_P, it follows that t_P is contained in π_P. Q.E.D.

Claim 2. *The transversal t_P does not coincide with m or n.*

Proof. The transversal t_P does not coincide with n because t_P is the span of P, a point on l, and N_P, a point on n, and l, n are skew lines. So t_P is the span of two points on two distinct lines, hence t_P does not coincide with either line. The transversal t_P does not coincide with m because m is skew to both l, n but t_P meets l, n in one point each. If t_P were to coincide with m, then m would meet l, n in one point each, which contradicts the skewness of l, m and n, m. Hence t_P does not coincide with m. Q.E.D.

Claim 3. *The transversal t_P meets both m and n.*

Proof. The transversal t_P meets n because t_P is the span of the points P and N_P, a point on the line n. Claim 2 tells us that t_P does not coincide with n, so t_P meets n at the point N_P only. To show that t_P meets m, it is sufficient to show that t_P and m both lie in the plane π_P. Projectively, coplanar lines either coincide or intersect. It again follows from Claim 2 that t_P does not coincide with m, so t_P meets m in one point. Q.E.D.

This proves the existence of the transversal t_P. The uniqueness of t_P is inherent in its construction: the point P is given and the point N_P is the unique point of intersection of the plane π_P (itself uniquely determined by P and the line m) and the line n. These two distinct points P, N_P contain a unique line, hence the transversal t_P is unique.

This gives us the following lemma:

Lemma 3.1.2. *Let l, m, n be three skew lines in* $\mathrm{PG}(3, \mathbb{R})$ *and let P be a point on l. Then there exists a unique transversal to l, m, n through the given point P.*

It is interesting to note that this problem appears as an exercise (3.1.5) in [9], which also indicates the link between transversals, reguli, and the two parallel classes of lines which generate the quadric:

> Let l, m, n be three *mutually skew* lines (i.e. no two of the lines intersect) of a projective space S_3 of dimension 3. Show that through each point of l, there exists a unique line r which intersects both m and n.
>
> Such a line r is called an (l, m, n)-**transversal**. The set $\mathfrak{R}$ of all (l, m, n)-transversals is called a **regulus**, and is sometimes denoted by $\mathfrak{R}(l, m, n)$. Prove that no two distinct (l, m, n)-transversals intersect in a point.

3.2 Proof of existence of a hyperbolic quadric

The aim of this section will be to prove that three skew lines and their regulus generate a unique quadric in $\mathrm{PG}(3, \mathbb{R})$. As will be seen in Section 4.1, there are, up to projective equivalence, two non-empty non-degenerate quadrics of $\mathrm{PG}(3, \mathbb{R})$, the hyperbolic and the elliptic. As aforementioned in Section 2.5, the hyperbolic contains lines, the elliptic does not. Hence a regulus and its complementary regulus generate a hyperbolic quadric. This is an important result, however in this section we emphasise that the form generated by three skew lines and their regulus is a quadratic form – a result which in itself is quite astounding.

We begin by showing that the action of $\mathrm{PGL}(4, \mathbb{R})$ on triples of skew lines is transitive (recall the definition of transitive actions in Section 2.6). Proving this will show that any triple of skew lines will be sent to a predetermined triple under the action of $\mathrm{PGL}(4, \mathbb{R})$.

Proposition 3.2.1. *The action of* $\mathrm{PGL}(4, \mathbb{R})$ *on triples of skew lines is transitive.*

Proof. In order to prove Proposition 3.2.1, we will first show that the action of $\mathrm{PGL}(4, \mathbb{R})$ is transitive on lines, then on pairs of skew lines, and finally, on triples of skew lines. This amounts to proving the following claims:

Claim 4. $\mathrm{PGL}(4, \mathbb{R})$ *is transitive on lines.*

Proof. We take $l_1 = \langle e_1, e_2 \rangle$ to be our 'first fundamental line'. We can represent l_1 as the row space of the 2×4 matrix $\begin{bmatrix} I & O \end{bmatrix}$, where I is the 2×2 identity matrix and O is the 2×2 zero matrix. Here representation by row space indicates that l_1 has the same number of non-zero rows as $\begin{bmatrix} I & O \end{bmatrix}$. An arbitrary line m_1 in $\mathrm{PG}(3, \mathbb{R})$ can be represented by the row space of the 2×4 matrix $\begin{bmatrix} A & B \end{bmatrix}$, where A, B are 2×2 matrices with full rank.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where C, D are any full rank 2×2 matrices. Then

$$\begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix},$$

so we can map l_1 to any line m_1 in $\mathrm{PG}(3, \mathbb{R})$ by an element of $\mathrm{PGL}(4, \mathbb{R})$. Clearly, there exists matrices C, D such that M is invertible. Hence $\mathrm{PGL}(4, \mathbb{R})$ is transitive on lines. Q.E.D.

Remark 3.2.2. *With notation as defined in Claim 4, let* $M_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}$*, then* M_1 *maps* $\begin{bmatrix} A & B \end{bmatrix}$ *to* $\begin{bmatrix} I & O \end{bmatrix}$*. This will be useful for the next claim.*

Claim 5. $\mathrm{PGL}(4, \mathbb{R})$ *is transitive on pairs of skew lines.*

Proof. Essentially, we will be showing that $\mathrm{PGL}(4, \mathbb{R})$ is transitive on pairs of skew lines. We will show this by proving that the stabiliser G_{l_1} of l_1 (as in Claim 4) in $\mathrm{PGL}(4, \mathbb{R})$ is transitive on lines skew to l_1.

Firstly, we construct a line l_2 that is skew to our first fundamental line l_1, that is, $l_1 \cap l_2 = \varnothing$. An equivalent condition is that l_1, l_2 span the whole space, which would mean that the matrix $\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ has full rank (here we are viewing lines as row spaces of matrices). If $l_2 = \begin{bmatrix} S & T \end{bmatrix}$, then the only necessary condition on l_2 is that T has rank 2. As such, let $l_2 = \begin{bmatrix} O & I \end{bmatrix}$ be our 'second fundamental line'.

Let $m_1 = \begin{bmatrix} A & B \end{bmatrix}$ as in Claim 4 and let $m_2 = \begin{bmatrix} C & D \end{bmatrix}$ be any line skew to m_1. Consider the matrix $M = \begin{bmatrix} A & O \\ C & D \end{bmatrix}$, where A, D are rank 2 matrices. Then

$$\begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} A & O \\ C & D \end{bmatrix} = \begin{bmatrix} A & O \end{bmatrix} \equiv \begin{bmatrix} I & O \end{bmatrix},$$

where $\equiv$ is used to indicate equality of row space. Hence M fixes l_1, so $M \in G_{l_1}$.

Moreover,

$$\begin{bmatrix} O & I \end{bmatrix} \begin{bmatrix} A & O \\ C & D \end{bmatrix} = \begin{bmatrix} C & D \end{bmatrix},$$

so M maps l_2 to any line skew to m_1. Let M_1 be as in Remark 3.2.2. Again, it is clear that M is invertible, so let $M_2 = M^{-1}$. Then the mapping M_1M_2 will map any pair of skew lines m_1, m_2 to the fundamental pair of skew lines, l_1, l_2: M_1 maps m_1 to l_1 and m_2 is mapped to m_2M_1 (which could be anything), then M_2 fixes l_1 and maps m_2M_1 to l_2. Hence $\mathrm{PGL}(4,\mathbb{R})$ is transitive on pairs of skew lines. Q.E.D.

Claim 6. $\mathrm{PGL}(4,\mathbb{R})$ *is transitive on triples of skew lines.*

Proof. We now consider triples of skew lines. We need to find a third fundamental line l_3 that is skew to both l_1 and l_2 (where l_1 and l_2 are defined as in Claim 5). The condition for three lines to be mutually skew is simply that all three lines are pairwise skew to each other. Some line $\begin{bmatrix} S & T \end{bmatrix}$ will be skew to l_1 if $\begin{bmatrix} I & O \\ S & T \end{bmatrix}$ has full rank. This is true when T has full rank. The line $\begin{bmatrix} S & T \end{bmatrix}$ will be skew to l_2 if $\begin{bmatrix} O & I \\ S & T \end{bmatrix}$ has full rank. This is true when S has full rank. Hence a line $\begin{bmatrix} S & T \end{bmatrix}$ will be skew to both l_1 and l_2 when both S and T have full rank. If we take $l_3 = \begin{bmatrix} I & I \end{bmatrix}$, it is clear that l_3 satisfies these conditions.

Let $m_1 = \begin{bmatrix} A & B \end{bmatrix}$ and $m_2 = \begin{bmatrix} C & D \end{bmatrix}$ be a pair of skew lines as in Claim 5 and let $m_3 = \begin{bmatrix} E & F \end{bmatrix}$ be such that m_1, m_2, m_3 form a triple of skew lines. Consider the matrix $M_3 = \begin{bmatrix} E^{-1} & O \\ O & F^{-1} \end{bmatrix}$. Now

$$\begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} E^{-1} & O \\ O & F^{-1} \end{bmatrix} = \begin{bmatrix} E^{-1} & O \end{bmatrix} \equiv \begin{bmatrix} I & O \end{bmatrix},$$

so M_3 fixes l_1. Consider also

$$\begin{bmatrix} O & I \end{bmatrix} \begin{bmatrix} E^{-1} & O \\ O & F^{-1} \end{bmatrix} = \begin{bmatrix} O & F^{-1} \end{bmatrix} \begin{bmatrix} O & I \end{bmatrix},$$

so M_3 also fixes l_2. Hence $M_3 \in G_{l_1,l_2}$, the stabiliser of l_1 and l_2. Then

$$\begin{bmatrix} E & F \end{bmatrix} \begin{bmatrix} E^{-1} & O \\ O & F^{-1} \end{bmatrix} = \begin{bmatrix} I & I \end{bmatrix},$$

so M_3 maps any line m_3 to l_3. Let M_1, M_2 be as in Claim 5. Then the mapping $M_1M_2M_3$ will map any three skew lines m_1, m_2, m_3 in $\mathrm{PG}(3,\mathbb{R})$ to the fundamental skew lines l_1, l_2, l_3: M_1 maps m_1 to l_1 and m_2, m_3 are mapped to m_2M_1, m_3M_1, respectively (these could be anything), then M_2 fixes l_1 and maps m_2M_1 to l_2 and m_3M_1 to $m_3M_1M_2$ (which still could be anything). Then M_3 fixes l_1, l_2 and maps $m_3M_1M_2$ to l_3. Hence $\mathrm{PGL}(4,\mathbb{R})$ is transitive on triples of skew lines. Q.E.D.

Together Claims 4, 5, and 6 tell us that the action of $\mathrm{PGL}(4,\mathbb{R})$ on triples of skew lines is transitive. Q.E.D.

Now we have shown transitivity on triples of skew lines, it remains to show that the triple of skew lines m_1, m_2, m_3 do indeed lie in a hyperbolic quadric (see Figure 3.3). To see this, we must first prove the following lemma:

Lemma 3.2.3. *Let M be the associated matrix of a symmetric bilinear form associated with the quadric $\mathcal{Q}$. Then a line m, written as the row space of a 2×4 matrix, lies in the quadric $\mathcal{Q}$ if and only if $mMm^\top$ is the zero matrix.*

Proof. Firstly, if a line m lies in a quadric $\mathcal{Q}$, then any point $P=(w,x,y,z)$ on m also lies in $\mathcal{Q}$. Hence $\mathcal{Q}(P)=0$. Let $\{e_1,\dots,e_4\}$ be the standard basis for $\mathbb{R}^4$, let B be the symmetric bilinear form associated with $\mathcal{Q}$, and let M be the matrix associated with $\mathcal{Q}$ with respect to the standard basis. Then it is clear by definition that $\begin{bmatrix}P\end{bmatrix} M \begin{bmatrix}P^\top\end{bmatrix}=0$ if and only if P is a point on $\mathcal{Q}$.

Now let $u=(w_1,x_1,y_1,z_1)$ and $v=(w_2,x_2,y_2,z_2)$ be points in $\mathrm{PG}(3,\mathbb{R})$ and let $m=\begin{bmatrix}u\\ v\end{bmatrix}$ be a line. Then

$$\begin{aligned} mMm^\top &= \begin{bmatrix}u\\ v\end{bmatrix} M \begin{bmatrix}u^\top & v^\top\end{bmatrix}\\ &= \begin{bmatrix}uM\\ vM\end{bmatrix}\begin{bmatrix}u^\top & v^\top\end{bmatrix}\\ &= \begin{bmatrix}uMu^\top & uMv^\top\\ vMu^\top & vMv^\top\end{bmatrix}. \end{aligned}$$

The entries $uMu^\top$ and $vMv^\top$ will be zero if and only if u, v are points on $\mathcal{Q}$, and then m is a line in the quadric $\mathcal{Q}$. Finally, the points u, v will be orthogonal with respect to the form $\mathcal{Q}$ if and only if they both lie in the quadric, in which case $uMv^\top = vMu^\top = 0$. Q.E.D.

With this lemma, we can prove the following:

Theorem 3.2.4. *Three mutually skew lines determine a unique (hyperbolic) quadric.*

Proof. Let $M=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$ be the associated matrix of a symmetric bilinear form with associated quadric $\mathcal{Q}$. Then M is by construction symmetric because M is the associated matrix of a symmetric bilinear form (Lemma 2.3.8), so $A=A^\top$, $D=D^\top$, and $C=B^\top$. Let l_1, l_2, l_3 be the fundamental triple of skew lines, as given in Proposition 3.2.1. Using Lemma 3.2.3, we compute:

$$l_1Ml_1^\top = \begin{bmatrix}I & O\end{bmatrix}\begin{bmatrix}A & B\\ B^\top & D\end{bmatrix}\begin{bmatrix}I & O\end{bmatrix}^\top = A,$$

so l_1 lies in a quadric when A is the 2×2 zero matrix. Similarly,

$$l_2 M l_2^\top = \begin{bmatrix} O & I \end{bmatrix} \begin{bmatrix} A & B \\ B^\top & D \end{bmatrix} \begin{bmatrix} O & I \end{bmatrix}^\top = D,$$

so l_2 lies in a quadric when D is the 2×2 zero matrix. Finally,

$$l_3 M l_3^\top = \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} A & B \\ B^\top & D \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix}^\top = B + B^\top,$$

so l_3 lies in a quadric when $B = -B^\top$. Hence l_1, l_2, l_3 lie in the quadric with associated matrix $M = \begin{bmatrix} O & B \\ -B^\top & O \end{bmatrix}$. The matrix B will be such that $B = -B^\top$ if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$. This is true when $a = d = 0$ and $b = -c$. Up to scalar equivalence (since we are working in $\mathrm{PGL}(4, \mathbb{R})$), the only matrix satisfying these conditions is $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Hence the lines l_1, l_2, l_3 lie in the quadric with associated matrix

$$M = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}.$$

Let m_1, m_2, m_3 be any three skew lines. Since we have shown in Proposition 3.2.1 that any three skew lines can be mapped to the fundamental triple of skew lines, there exists some matrix $H \in \mathrm{PGL}(4, \mathbb{R})$ such that $m_1^H = l_1$, $m_2^H = l_2$, and $m_3^H = l_3$. Then the associated matrix of the unique quadric determined by m_1, m_2, m_3 is given by $HMH^\top$. Q.E.D.

Remark 3.2.5. *Most proofs of the generation of a hyperbolic quadric from three skew lines are analytic, in the sense that they show that the parameters of the lines 'fit' a quadratic form (see for example [5], Theorem 2.4.4, [18], Section 81.1, and [22], Chapter XI). The proof we have just provided is novel in that the result is derived from the symmetry of the form.*

3.3 Proof of skew projection

In this section, we tie together previous results from this chapter and prove that skew projection does indeed generate a conic. Firstly, we need to establish some more results about quadrics and skew lines: that a hyperbolic quadric intersects planes in $\mathrm{PG}(3, \mathbb{R})$ in a conic and that the transversals to three skew lines do not intersect.

Proposition 3.3.1. *A hyperbolic quadric will intersect a given plane in a conic.*

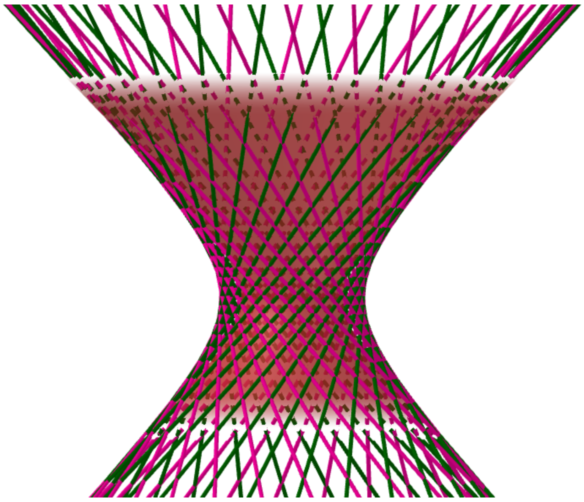

Figure 3.3: A hyperbolic quadric. Note that the two parallel classes of lines which comprise the hyperbolic quadric are clearly visible as pink and green lines.

Proof. Let V be a vector space and let B be a symmetric bilinear form associated with a quadric $\mathcal{Q}$. Let π be a plane, then π is defined by three vectors, that is, $\pi = \langle u_1, u_2, u_3 \rangle$ for some vectors u_1, u_2, u_3. Let v be a vector other than u_1, u_2, u_3 (such a v exists), then $\mathcal{B} = \{u_1, u_2, u_3, v\}$ defines a basis for V. Then the associated matrix of $\mathcal{Q}$ with respect to $\mathcal{B}$ is $M_{\mathcal{B}} = \begin{bmatrix} N & X \\ Y & Z \end{bmatrix}$, a symmetric matrix. If we restrict $M_{\mathcal{B}}$ to the plane π, we have $M_{\mathcal{B}}|_\pi = N$ is symmetric, so by Lemma 2.3.8, $M_{\mathcal{B}}|_\pi$ must be the associated matrix of a (possibly degenerate) symmetric bilinear form. Hence a quadric $\mathcal{Q}$ intersects a plane π in a conic. Q.E.D.

Proposition 3.3.2. *The transversals to three skew lines do not meet.*

Proof. Let l, m, n be three skew lines in $\mathrm{PG}(3, \mathbb{R})$ and suppose t_1, t_2 are transversals to l, m, n through the points P_1, P_2, respectively, on l such that t_1 and t_2 meet in a point. By the Grassmann Identity (2.1), we see that

$$\begin{aligned} \dim\langle t_1, t_2 \rangle &= \dim(t_1) + \dim(t_2) - \dim(t_1 \cap t_2) \\ &= 1 + 1 - 0 \\ &= 2, \end{aligned}$$

so t_1, t_2 span a plane. If this is so, then it follows that l, m, n are coplanar, which is a contradiction. Hence the transversals to three skew lines do not meet. Q.E.D.

With these results, we can now prove that skew projection generates a conic (see Figure 3.4).

Theorem 3.3.3. *Skew projection generates a conic, that is, for three skew lines l, m, n, a point P on l, a transversal t_P to l, m, n through P, and a plane π, the set $\{t_P \cap \pi : P \cap l\}$ defines a conic.*

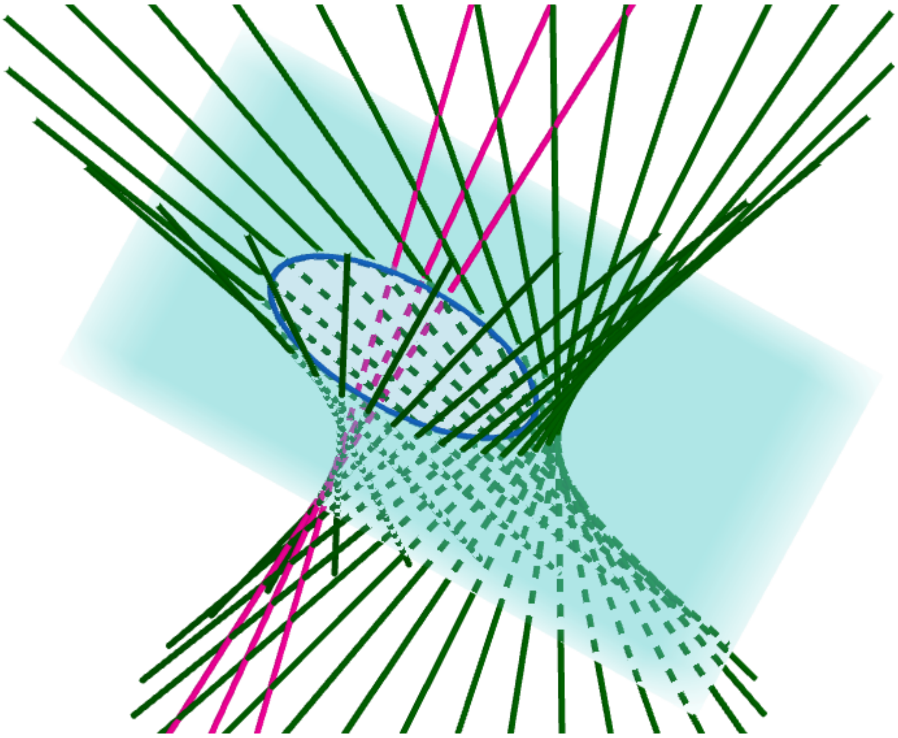

Figure 3.4: A view of skew projection generating a conic. Here, the pink lines are the three skew lines l, m, n and the green lines form the regulus to l, m, n.

Proof. Let l, m, n be three skew lines, let π be a plane, and let P, P', P'' be points on l. Let t_P, $t_{P'}$, $t_{P''}$ be the transversals to l, m, n through P, P', P'', respectively. By Proposition 3.3.2, t_P, $t_{P'}$, $t_{P''}$ are skew, so by Theorem 3.2.4, they determine a unique hyperbolic quadric. By Proposition 3.3.1, this quadric defines a unique conic on the plane π. Since this conic is unique, the transversal t_S to l, m, n through $S \neq P$, P', P'' lies in the same quadric as t_P, $t_{P'}$, $t_{P''}$. Since transversals are unique for every point on l, this gives us a one-to-one correspondence between the points on l and the points on the conic. Q.E.D.

Chapter 4

Classifications

In the previous chapter we proved that skew projection generates a conic by eliciting the relationship between skew projection and hyperbolic quadrics. In this chapter, we continue our investigation into hyperbolic quadrics. As such, we will classify the non-empty non-degenerate quadrics in $\mathrm{PG}(3, \mathbb{R})$ via Sylvester's Theorem (4.1.2). We will then determine which conics are generated by skew projection by considering the orbits of the quadric on degenerate and non-degenerate planes via Witt's Theorem (4.3.3). Finally, we determine the equivalent conditions on three skew lines for generating these conics via a skew projection.

4.1 Classifying projective quadrics

The aim of this section will be to classify the projective quadrics in $\mathrm{PG}(3, \mathbb{R})$. The essence of this classification is captured well by this quote, found in [4] (Definition 13.1.4.1):

> The classification of quadratic forms over a field $\mathbb{K}$ is the problem of finding the equivalence classes of quadratic forms on finite-dimensional vector spaces over $\mathbb{K}$.

This will be achieved by proving the following theorem:

Theorem 4.1.1. *Up to projective equivalence, there are two non-degenerate non-empty quadrics of* $\mathrm{PG}(3, \mathbb{R})$,

1. *(elliptic)* $w^2 + x^2 + y^2 - z^2 = 0$ *and*
2. *(hyperbolic)* $w^2 + x^2 - y^2 - z^2 = 0$.

We will classify the affine quadrics in $\mathbb{R}^3$ in Section 4.2. There are many more affine quadrics than projective quadrics and Theorem 4.1.1 is a beautiful example of the simplicity of projective geometry. With respect to skew projection, Theorem 4.1.1 asserts that the quadrics of $\mathrm{PG}(3, \mathbb{R})$ can be categorised into those con-

Figure 4.1: James Joseph Sylvester (1814-1897) [17].

taining lines (hyperbolic) and those which do not contain lines (elliptic) and so the reguli which comprise skew projection generate a hyperbolic quadric.

Sylvester's Law of Inertia

James Josephs Sylvester (1814-1897) was a British mathematician, specialising as an algebraist and especially known for his work on problems in number theory. Although placing second in the mathematical tripos at the University of Cambridge in 1837, Sylvester was unable to graduate or receive an appointment – this was reserved for members of the Church of England and Sylvester was of the Jewish faith. In spite of this, what followed for Sylvester was an illustrious career in mathematics (and other areas). In 1838 he was appointed as a professor of natural philosophy at University College, London and later received an appointment at the University of Virginia. In 1839 he was made a fellow of the Royal Society and he was the second president of the London Mathematical Society from 1866-68. In 1843 he returned to England and took up actuarial studies and later legal studies, being admitted to the bar in 1846. It was as a lawyer that he met another brilliant English mathematician, Arthur Cayley, with whom he collaborated on mathematics. In 1855 he was appointed as a professor of mathematics at the Royal Military Academy, Woolwich, and in 1876 he was given a professorship at John Hopkins University. There he made a profound contribution to American mathematics, founding and editing the American Journal of Mathematics. In 1883 he was made the Savilian Professor of Geometry at the University of Oxford. He also wrote poetry [8].

Our classification of projective quadrics will require an alternative formulation of Sylvester's Law of Inertia. The following formulation of Sylvester's Law is derived from [2], Theorem 2.11, and [21], Theorem 11.21. See also the commentary in [1], Chapter III, Section 7.

Theorem 4.1.2 (Sylvester's Law of Inertia)**.** *Let V be an n-dimensional real vector space and let B be a symmetric bilinear form on V.*

1. *Then there is an orthogonal basis $\mathcal{B} = (v_1, \ldots, v_n)$ for V such that $B(v_i, v_j) = 0$ if $i \neq j$ and for each i, $B(v_i, v_i)$ is equal to 1, -1, or 0. The basis $\mathcal{B}$ can be ordered such that $\mathcal{B} = (s_1, \ldots, s_p, t_1, \ldots, t_m, u_1, \ldots, u_z)$, where $p + m + z = n$ and $B(s_i, s_i) = 1$, $B(t_i, t_i) = -1$, and $B(u_i, u_i) = 0$. Hence the form B can be represented by the diagonal matrix*

$$M_{\mathcal{B}} = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix}.$$

2. *The numbers p, m, and z are uniquely determined by the form B and are not dependent on the choice of orthogonal basis $\mathcal{B}$. The pair (p, m) is called the* signatue *of the form.*

For clarity, here is an example in the low-dimensional case $n = 2$:

Example 4.1.3. *Let $V = \mathbb{R}^2$, a 2-dimensional real vector space, and let $\mathcal{B} = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$, that is $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then we have three unique non-empty bilinear forms on V with respect to $\mathcal{B}$:*

$${}^1M_{\mathcal{B}} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \; {}^2M_{\mathcal{B}} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \; {}^3M_{\mathcal{B}} = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}.$$

These correspond to the bilinear forms $x^2 + y^2 = 0$, $x^2 - y^2 = 0$, and $x^2 = 0$.

Before proving this theorem, we will need to prove two simple propositions:

Proposition 4.1.4. *Suppose there is a symmetric bilinear form B on a vector space V that is not identically zero. Then there is a vector $v \in V$ which is not self-orthogonal, that is, $B(v, v) \neq 0$.*

Proof. If B is not identically zero, then there are some vectors v, $w \in V$ such that $B(v, w) \neq 0$. If either $B(v, v) \neq 0$ or $B(w, w) \neq 0$, we are done. If not, let $u = v + w$. Note that $u \in V$ because V is a vector space. We have

$$\begin{aligned} B(u, u) &= B(v + w, v + w) \\ &= B(v, v) + B(v, w) + B(w, v) + B(w, w) \\ &= B(v, v) + 2B(v, w) + B(w, w), \end{aligned}$$

by symmetry, and since $B(v,v) = B(w,w) = 0$ by assumption, we have

$$B(v,v) + 2B(v,w) + B(w,w) = 2B(v,w) \neq 0$$

Hence, the vector u is not self-orthogonal. Q.E.D.

Proposition 4.1.5. *Let $w \in V$ be a vector which is not self-orthogonal. Let the span of w be W. Then V is the direct sum of W and its orthogonal complement, that is,*

$$V = W \oplus W^{\perp},$$

where $W^{\perp} = \{v \in V : B(w,v) = 0$ for every $w \in W\}$.

In order to prove this proposition, it is sufficient to show that W and $W^{\perp}$ have empty intersection and that the sum of W, $W^{\perp}$ spans V. Then the result follows because W, $W^{\perp}$ satisfy the requisite properties for *direct sums of vector subspaces*, that is, if we have some subspaces $S_1, \ldots, S_k$ of V, then V is the direct sum of the subspaces S_i if $V = \sum_{i=1}^{k} S_i$ and for each $i \in \{1, \ldots, k\}$, $S_i \cap (\sum_{i \neq j} S_j) = \{0\}$ ([21], Chapter 1). Our proof of Proposition 4.1.5 follows [2] (Proposition 2.4 in this text).

Proof. Firstly, we show that W, $W^{\perp}$ have empty intersection. As such, let $cw \in W$. If $cw \in W^{\perp}$, then $B(cw, dw) = 0$ for any $dw \in W$. But $B(cw, dw) = cdB(w,w)$ (by linearity) and $B(w,w) \neq 0$ by assumption.

Next, we show that W, $W^{\perp}$ span V. We will achieve this by showing that any $v \in V$ can be written as a sum of vectors in W, $W^{\perp}$, that is $v = aw + w'$, $a \in V$, $w \in W$, $w' \in W^{\perp}$.

We solve $B(v - aw, w) = 0$ for a in order to choose a cleverly:

$$\begin{aligned} B(v - aw, w) &= 0 \\ \Leftrightarrow B(v,w) - B(aw,w) &= 0 \\ \Leftrightarrow B(v,w) - aB(w,w) &= 0. \end{aligned}$$

The previous two lines are due to the linearity of the bilinear operator. From these calculations, we find $a = \frac{B(v,w)}{B(w,w)}$, which is defined since $B(w,w) \neq 0$.

Set $w' = v - aw$, with a defined as above. It remains to show that $B(w,w') = 0$ (so that indeed $w' \in W^{\perp}$) and $v = aw + w'$. The second equality is immediate. For the first equality we compute:

$$\begin{aligned} B(w,w') &= B(w, v - aw) \\ &= B(w,v) - B(w,aw) \\ &= B(w,v) - aB(w,w) \\ &= B(w,v) - \frac{B(v,w)}{B(w,w)} B(w,w) \\ &= B(w,v) - B(v,w) \\ &= 0, \end{aligned}$$

where the second and third lines of calculation are due to the linearity of B and the final line is due to the symmetry of B.

Therefore, $W \cap W^{\perp} = \{0\}$ and $V = W + W^{\perp}$, so it follows from the definition of direct sums of vector subspaces that $V = W \oplus W^{\perp}$. Q.E.D.

We are now ready to prove Theorem 4.1.2, following the proof found in [2] (Theorem 2.11 in this text).

Proof of Theorem 4.1.2. We first prove that we can represent the form B by the matrix $M_{\mathcal{B}}$. If B is identically zero, then the associated matrix $M_{\mathcal{B}}$ will comprise only zero entries with respect to any basis and hence $M_{\mathcal{B}}$ will be a diagonal matrix with respect to any basis, as required.

Now suppose B is not identically zero. Then by Proposition 4.1.4, there is a vector v_j, $j \leqslant n$, such that $B(v_j, v_j) \neq 0$, in other words, because B is not identically zero, there must be a nonzero vector on the diagonal of $M_{\mathcal{B}}$. Let W be the span of v_j. Then by Proposition 4.1.5, $V = W \oplus W^{\perp}$. Without loss of generality, let $v_j = v_1$ and by renumbering, let $(w_2, \ldots, w_n)$ be a basis for $W^{\perp}$. Then a basis for V will be $(v_1, w_2, \ldots, w_n)$.

By restricting B to $W^{\perp}$, we have a form on $W^{\perp}$ because if B is a form on V, it must certainly be a form on the subspace $W^{\perp}$. Hence we can orthogonalise the vectors $w_2, \ldots, w_n$ with respect to B and in consequence we obtain orthogonal basis vectors $(v_2, \ldots, v_n)$ for $W^{\perp}$.

Since v_1 is by definition orthogonal to every element of $W^{\perp}$, we have $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ is an orthogonal basis for V.

It now remains to normalise $\mathcal{B}$. If $B(v_i, v_i) = 0$, we are done. If $B(v_i, v_i) \neq 0$, then set $c^{-2} = \pm B(v_i, v_i)$, replace v_i with cv_i, and note that such a c exists. Then

$$B(cv_i, cv_i) = cB(v_i, cv_i) = c^2 B(v_i, v_i) = \frac{B(v_i, v_i)}{B(v_i, v_i)} = \pm 1.$$

We can now permute the basis vectors as necessary to have $M_{\mathcal{B}}$ in the required form.

The second step is to show that p, m, and z are uniquely determined by the form. As such, notice that $M_{\mathcal{B}}$ is a diagonal matrix in reduced row echelon form and hence $p+m$ is the rank of $M_{\mathcal{B}}$, since z is the number of zero rows in the reduced form. With the basis $\mathcal{B}$ orthogonal and normalised, we show that $(v_{p+m+1}, \ldots, v_n)$ form a basis for the null space N of V in order to show that z is determined by B.

If a vector $w \in V$ is in the null space N, then $B(w, v_i) = 0$ for every $v_i \in \mathcal{B}$ (in fact, the converse is also true). Since $w \in V$, we can write w as a linear combination of basis vectors $w = c_1 v_1 + \ldots + c_n v_n$, for some $\{c_i\} \in V$. Then

$$\begin{aligned} B(w, v_i) &= B(c_1 v_1 + \ldots + c_i v_i + \ldots + c_n v_n, v_i) \\ &= B(c_1 v_1, v_i) + \ldots + B(c_i v_i, v_i) + \ldots + B(c_n v_n, v_i), \end{aligned}$$

since B is bilinear. Note that since the vectors v_i are orthogonal basis vectors, $B(v_i, v_j) = 0$ if $i \neq j$, so

$$B(c_1 v_1, v_i) + \ldots + B(c_i v_i, v_i) + \ldots + B(c_n v_n, v_i) = B(c_i v_i, v_i)$$

$$= c_i B(v_i, v_i),$$

again, since B is bilinear.

$B(v_i, v_i)$ will be nonzero when $i \leqslant p+m$, so c_i must be zero for all $i \leqslant p+m$ in order for w to be orthogonal to v_i as proposed. So $w = c_{p+m+1}v_{p+m+1} + \ldots + c_n v_n$. The vector $w \in N$ was arbitrary, so we can represent every element in the null space by the vectors $(v_{p+m+1}, \ldots, v_n)$. Hence $(v_{p+m+1}, \ldots, v_n)$ is a linearly independent set spanning N, so it forms a basis for N. Now $(v_{p+m+1}, \ldots, v_n)$ are the z self-orthogonal vectors, so it follows that $z = \dim(N)$ and z is determined by the form.

Since $p+m$ satisfies $p+m+z = n$, $p+m$ is determined since z is determined. It remains to show that either one of p, m is uniquely determined. Without loss of generality, we will show that p is determined.

Suppose that from a second orthogonal and normalised basis $\mathcal{B}' = (v'_1, \ldots, v'_n)$ we obtain integers p', m', such that $p'+m'+z = n$. Consider the $p+(n-p')$ vectors $(v_1, \ldots, v_p, v'_{p'+1}, \ldots, v'_n)$. We want to show that these vectors are linearly independent. Suppose not, then there exists scalars $\{b_i\}$, $\{c_i\}$ in V such that

$$b_1 v_1 + \ldots + b_p v_p = c_{p'+1} v'_{p'+1} + \ldots + c_n v'_n.$$

Let $v = b_1 v_1 + \ldots + b_p v_p = c_{p'+1} v'_{p'+1} + \ldots + c_n v'_n$. Then

$$\begin{aligned} B(v, v) &= B(b_1 v_1 + \ldots + b_p v_p, b_1 v_1 + \ldots + b_p v_p) \\ &= B(b_1 v_1, b_1 v_1) + \ldots + B(b_p v_p, b_p v_p) \\ &= b_1^2 B(v_1, v_1) + \ldots + b_p^2 B(v_p, v_p), \end{aligned}$$

where the previous two lines are due to the linearity of B. Now, since $B(v_i, v_i) = 1$ for $i \leqslant p$, we have

$$b_1^2 B(v_1, v_1) + \ldots + b_p^2 B(v_p, v_p) = b_1^2 + \ldots + b_p^2 \geqslant 0.$$

On the other hand,

$$\begin{aligned} B(v, v) &= B(c_{p'+1} v'_{p'+1} + \ldots + c_n v'_n, c_{p'+1} v'_{p'+1} + \ldots + c_n v'_n) \\ &= B(c_{p'+1} v'_{p'+1}, c_{p'+1} v'_{p'+1}) + \ldots + B(c_n v'_n, c_n v'_n) \\ &= c_{p'+1}^2 B(v'_{p'+1}, v'_{p'+1}) + \ldots + c_n^2 B(v'_n, v'_n), \end{aligned}$$

where the previous two lines are due to the linearity of B. Now, since $B(v_i, v_i) = -1$ for $p' < i \leqslant m'$ and $B(v_i, v_i) = 0$ for $m' < i \leqslant n$, we have

$$c_{p'+1}^2 B(v'_{p'+1}, v'_{p'+1}) + \ldots + c_n^2 B(v'_n, v'_n) = -c_{p'+1}^2 - \ldots - c_{p'+m'}^2 \leqslant 0.$$

Hence $0 \leqslant B(v, v) \leqslant 0$, so $B(v, v) = 0 = b_1^2 + \ldots + b_p^2$ and in consequence, $b_1 = \ldots = b_p = 0$. Since $b_1 = \ldots = b_p = 0$, we have $0 = b_1 v_1 + \ldots + b_p v_p = c_{p'+1} v'_{p'+1} + \ldots + c_n v'_n$. But $(v'_1, \ldots, v'_n)$ is a basis, so if $c_{p'+1} v'_{p'+1} + \ldots + c_n v'_n = 0$, then $c_{p'+1} = \ldots = c_n = 0$.

Hence the linear relation between the $p+(n-p')$ vectors is trivial and they are linearly independent. As a result, $p+(n-p') \leqslant n$, so $p \leqslant p'$. By interchanging the roles of p and p', we obtain $p' \leqslant p$, so $p = p'$. Hence p is determined and so m is determined. Q.E.D.

Let $\mathcal{B}$ be a basis as in Theorem 4.1.2. Then for $x \in \mathrm{PG}(n,\mathbb{R})$, we have

$$xM_Bx^\top = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{m} x_i^2 = B(x,x).$$

Let g be the collineation taking any basis $\mathcal{B}'$ to $\mathcal{B}$ – such a g exists because change of basis is a linear map. This gives us an alternative formulation of Theorem 4.1.2, from which the proof of Theorem 4.1.1 will follow. We state this alternative formulation as the following theorem:

Theorem 4.1.6. *Let $\mathcal{Q}$ be a quadric of $\mathrm{PG}(n,\mathbb{R})$. Then there is a collineation g of $\mathrm{PG}(n,\mathbb{R})$ and integers l, m in $\{1,\dots,n+1\}$ such that $l+m \leqslant n+1$, $l \leqslant m$, and $\mathcal{Q}^g$ has quadratic form*

$$x_1^2 + \ldots + x_m^2 - (x_{m+1}^2 + \ldots + x_{m+l}^2) = 0.$$

Using Theorem 4.1.6 above, we now prove Theorem 4.1.1:

Proof of Theorem 4.1.1. Let $\mathcal{Q}$ be a quadric of $\mathrm{PG}(3,\mathbb{R})$. Then $\mathcal{Q}$ satisfies the conditions of Theorem 4.1.6 with $n=3$, so we have three cases.

Case 1. $l=0$, $m=4$: There is a collineation g_1 of $\mathrm{PG}(3,\mathbb{R})$ such that $\mathcal{Q}^{g_1}$ has quadratic form

$$w^2 + x^2 + y^2 + z^2 = 0,$$

which has no real solution in $\mathrm{PG}(3,\mathbb{R})$ because the point $(0,0,0,0) \notin PG(3,\mathbb{R})$. Then the first case is the empty quadric in $\mathrm{PG}(3,\mathbb{R})$.

Case 2. $l=1$, $m=3$: There is a collineation g_2 of $\mathrm{PG}(3,\mathbb{R})$ such that $\mathcal{Q}^{g_2}$ has quadratic form

$$w^2 + x^2 + y^2 - z^2 = 0,$$

the elliptic quadric of $\mathrm{PG}(3,\mathbb{R})$.

Case 3. $l=2$, $m=2$: There is a collineation g_3 of $\mathrm{PG}(3,\mathbb{R})$ such that $\mathcal{Q}^{g_3}$ has quadratic form

$$w^2 + x^2 - y^2 - z^2 = 0,$$

the hyperbolic quadric of $\mathrm{PG}(3,\mathbb{R})$.

Hence all quadrics of $\mathrm{PG}(3,\mathbb{R})$ can be transformed via a collineation to the empty, elliptic, or hyperbolic quadric. Thus the elliptic and hyperbolic quadrics are the only non-degenerate non-empty quadrics of $\mathrm{PG}(3,\mathbb{R})$ up to projective equivalence. Q.E.D.

Remark 4.1.7. *Note that the hyperbolic quadric $w^2+x^2-y^2-z^2=0$ has the same eigenvalues as the hyperbolic quadric found in Theorem 3.2.4 and presented in Example 2.3.3. To see this, let*

$$A = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}, \; B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \; \textit{and } C = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Let $\lambda \in \mathbb{R}$ and I be the 4×4 identity matrix. Then $det(A - \lambda I) = det(B - \lambda I) = det(C - \lambda I) = (x-1)^2(x+1)^2$, *so A, B, and C have the same eigenvalues $\lambda = \pm 1$. The matrices A, B, and C are the associated matrices of quadratic forms with respect to some bases $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$. Since collineations preserve eigenvalues, there exists collineations g_1 mapping $\mathcal{B}_1$ to $\mathcal{B}_2$ and g_2 mapping $\mathcal{B}_3$ to $\mathcal{B}_2$ (the collineations g_1, g_2 are simply change of basis matrices). Hence, the matrices A, B, and C are each the associated matrix of the unique quadratic form with signature* $(2, 2)$.

4.2 Classifying affine quadrics

In Section 4.1, we classified the projective quadrics. We now provide a classification of affine quadrics. Often in affine geometry it is necessary to work through case analyses which can be rather tedious, whereas projectively there is only one case. Moreover, we can move between affine and projective geometries by adding or removing the line at infinity. The combination of there being less quadrics projectively and the ability to link back to the affine case makes projective geometry particularly interesting.

It is only necessary to classify the projective quadrics in order to answer our questions about skew projection. However, out of interest, in this section we classify affine quadrics. In order to classify affine quadrics, we first prove the following proposition (a variant of this proposition can be found in [4], Proposition 15.3.2):

Proposition 4.2.1. *Let Q be a quadratic form over $\mathbb{R}^n$ with associated matrix M_B for a fixed basis $\mathcal{B}$. Then the orbits of Q under the action of* $\mathrm{GL}(n, \mathbb{R})$ *are represented by the following forms:*

1. $\mathcal{Q}_1(p,m)$: $\sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+m} x_i^2$; $\quad p \geqslant m,\ 1 \leqslant p+m \leqslant n$,
2. $\mathcal{Q}_2(p,m)$: $\sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+m} x_i^2 + 1$; $\quad 1 \leqslant p+m \leqslant n$,
3. $\mathcal{Q}_3(p,m)$: $\sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+m} x_i^2 + 2x_n$; $\quad p \geqslant m,\ 1 \leqslant p \leqslant n-1$.

Proof. The proposition comes as a result of Theorems 4.1.2 and 4.1.6. Q.E.D.

With Proposition 4.2.1 proven, we can now proceed to prove the following theorem, which classifies the affine quadrics of $\mathbb{R}^3$:

Theorem 4.2.2. *There are five non-empty non-degenerate quadrics of $\mathbb{R}^3$, namely*

1. *(ellipsoid)* $-x^2 - y^2 - z^2 + 1 = 0$,
2. *(one-sheet hyperboloid)* $x^2 - y^2 - z^2 + 1 = 0$,
3. *(two-sheet hyperboloid)* $x^2 + y^2 - z^2 + 1 = 0$,
4. *(hyperbolic paraboloid)* $x^2 - y^2 + 2z = 0$, *and*
5. *(elliptic paraboloid)* $x^2 + y^2 + 2z = 0$.

Proof. Using Proposition 4.2.1 with $n = 3$, we have to test the following pairs (p, m):

1. $\mathcal{Q}_1(p,m)$: $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(3,0)$;
2. $\mathcal{Q}_2(p,m)$: $(0,1)$, $(0,2)$, $(0,3)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(3,0)$;
3. $\mathcal{Q}_3(p,m)$: $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$.

Rather tediously, we must now compute $\mathcal{Q}_1(p,m)$, $\mathcal{Q}_2(p,m)$, and $\mathcal{Q}_3(p,m)$ for all the pairs (p,m).

$\mathcal{Q}_1(1,0)$: $x^2 = 0$, is the plane $x = 0$. $\mathcal{Q}_1(1,1)$: $x^2 - y^2 = 0$, is the union of two planes $x - y = 0$ and $x + y = 0$. $\mathcal{Q}_1(2,0)$: $x^2 + y^2 = 0$, is the line $\{(0,0,z) : z \in \mathbb{R}\}$. $\mathcal{Q}_1(2,1)$: $x^2 + y^2 - z^2 = 0$, is the degenerate cone with vertex $(0,0,0)$. $\mathcal{Q}_1(3,0)$: $x^2 + y^2 + z^2 = 0$, is the point $(0,0,0)$.

$\mathcal{Q}_2(0,1)$: $-x^2+1 = 0$, is the union of two parallel planes $-x+1 = 0$ and $x+1 = 0$. $\mathcal{Q}_2(0,2)$: $-x^2 - y^2 + 1 = 0$, is an infinite cylinder. $\mathcal{Q}_2(0,3)$: $-x^2 - y^2 - z^2 + 1 = 0$, is an ellipsoid. $\mathcal{Q}_2(1,0)$: $x^2 + 1 = 0$, is the empty set. $\mathcal{Q}_2(1,1)$: $x^2 - y^2 + 1 = 0$, is two curved sheets. $\mathcal{Q}_2(1,2)$: $x^2 - y^2 - z^2 + 1 = 0$ is the one-sheet hyperboloid. $\mathcal{Q}_2(2,0)$: $x^2+y^2+1 = 0$, is the empty set. $\mathcal{Q}_2(2,1)$: $x^2+y^2-z^2+1$, is the two-sheet hyperboloid. $\mathcal{Q}_2(3,0)$: $x^2 + y^2 + z^2 + 1 = 0$, is the empty set.

$\mathcal{Q}_3(1,0)$: $x^2 + 2z = 0$, is a curved sheet. $\mathcal{Q}_3(1,1)$: $x^2 - y^2 + 2z = 0$, is the hyperbolic paraboloid. $\mathcal{Q}_3(2,0)$: $x^2+y^2+2z = 0$, is the elliptic paraboloid. $\mathcal{Q}_3(2,1)$: $x^2 + y^2 - z^2 + 2z = 0$, is simply a scaling of $\mathcal{Q}_2(2,1)$. Q.E.D.

4.3 Determining the conics

In this section, we will determine the orbits of the stabiliser of the hyperbolic quadric on planes, recognising that the conic generated by skew projection is the intersection of a plane with a hyperbolic quadric (as was shown in Section 3.2). We will begin by stating Witt's Theorem, a necessary theorem for classifying the orbits on planes, and we will provide a sketch of the proof. We will proceed to determine the orbits of the hyperbolic quadric on planes, which will answer the question of how the different conics are generated by skew projection.

Witt's Theorem

Witt's Theorem will be necessary to compute the orbits of the quadrics on planes (Section 4.3). The theorem is an extension theorem, it tells us that if we have a form on a subspace, then it can be extended to the whole space. This will be useful because it tells us that if we have a particular kind of subspace (a particular plane) with a particular form on it (the associated bilinear form of the quadric restricted to the plane), then all conics which satisfy the form are equivalent because they will be in the same orbit.

Witt's Theorem was first introduced by Ernst Witt (1911-1991) in his paper *Theorie der quadratischen Formen in beliebigen Körpern* (1937) as a 'cancellation theorem' [28]. Its equivalence to the extension theorem is noted in a later remark. We reproduce this theorem and remark for the reader's interest (the statements given in Theorem 4.3.1 and Theorem 4.3.2 below are similar):

> **Satz 4.** *Aus $\mathfrak{R}_1 + \mathfrak{R}_3 \cong \mathfrak{R}_2 + \mathfrak{R}_3$ darf $\mathfrak{R}_1 \cong \mathfrak{R}_2$ geschlossen werden.*
>
> ...
>
> **Anmerkung.** Aus Satz 4 können leicht folgende Tatsachen erschlossen werden: Jede Lösung ω_{i1} der Gleichung $\sum_i a_i x_i^2 = a_1$ läßt sich

zu einer Substitution $x_i = \sum_i \omega_{ik} y_k$ ergänzen, die Form $\sum_{a_i} x_i^2$ festläßt. Ebenfalls jede Lösung w_{i1}, ω_{i2} des Gleichungssystems

$\sum_i a_i x_{i1}^2 = a_1$, $\sum_i a_i x_{i2}^2 = a_2$, $\sum_i a_i x_{i1} x_{i2} = 0$. Usw.

Sind f und g zwei quadratische Formen, und sind die Variablen der einen Form unabhängig von den Variablen der anderen, so bilden wir die Summe $f + g$. Satz 4 können wir dann auch so aussprechen:

Aus $f_1 + f_3 \cong f_2 + f_3$ *darf* $f_1 \cong f_2$ *gleschlossen werden.*

It is beyond the scope of this thesis to provide a detailed proof of the theorem, however, an outline of the proof will be provided and more detailed proofs can be found in [4] (Theorem 13.7.1), [10], and [26] (Theorem 7.4). Before citing and proving the theorem, however, it will be necessary to briefly introduce some terminology.

For a vector space V, we have already seen that the orthogonal complement $V^\perp$ of V is the set of those elements of V which are orthogonal to every other element of V with respect to a bilinear form B, that is

$$V^\perp = \{u \in V : B(u, v) = 0 \text{ for every } v \in V\}.$$

Furthermore, the *radical* $\mathrm{rad}(V)$ of V is the set of all degenerate vectors in V, where a vector $w \in V$ is said to be degenerate with respect to a bilinear form B if $B(w, v) = 0$ for every $v \in V$. Clearly for V a vector space, $\mathrm{rad}(V) = V^\perp$. However, if $S \subseteq V$, then $\mathrm{rad}(S)$ is the set of all degenerate vectors in S, whereas $S^\perp$ is the set of all vectors in V which are orthogonal to S, hence $\mathrm{rad}(S) = S \cap S^\perp$ ([21], see note after Theorem 11.3). Hence we make the distinction between radicals and orthogonal complements in the proof of Theorem 4.3.3.

A vector space V is said to be *nonsingular* if its radical is trivial, that is, $\mathrm{rad}(V) = \{0\}$. Let V, W be vector spaces over a field $\mathbb{F}$ and let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quadratic forms on V, W respectively. The map $\sigma : (V, \mathcal{Q}_1) \to (W, \mathcal{Q}_2)$ is an isometry if it preserves quadratic forms, in the sense that for every $v \in V$, $\mathcal{Q}_2(\sigma(v)) = \mathcal{Q}_1(v)$. A hyperbolic pair is a pair of vectors (u, v) such that u, v are self-orthogonal with respect to B and $B(u, v) = 1$.

Here are the cancellation and extension theorems as they appear in [10] (appearing as Theorems 7.1 and 7.2 in this work):

Theorem 4.3.1 (Witt's Cancellation Theorem)**.** *Let U_1, U_2, V_1, V_2 be quadratic spaces (that is, vector spaces with a quadratic form), with V_1 and V_2 isometric. If $U_1 \oplus V_1 \cong U_2 \oplus V_2$, then $U_1 \cong U_2$.*

Theorem 4.3.2 (Witt's Extension Theorem)**.** *Let X_1 and X_2 be isometric quadratic spaces. Suppose we are given orthogonal direct sum decompositions $X_1 = U_1 \oplus V_1$, $X_2 = U_2 \oplus V_2$ and an isometry $f : V_1 \to V_2$. Then there exists an isometry $F : X_1 \to X_2$ such that $F|_{V_1} = f$ and $F(U_1) = U_2$.*

We state the theorems in the form of Theorems 4.3.1 and 4.3.2 because a straightforward explanation of the equivalence of the theorems can be found in [10], which we now follow. Assuming the extension theorem, suppose we have quadratic spaces U_1, U_2, V_1, V_2 satisfying the conditions of the cancellation theorem. Let $X_1 = U_1 \oplus V_1$, $X_2 = U_2 \oplus V_2$, and $f : V_1 \to V_2$ be an isometry (such an isometry does exist because we assumed that V_1, V_2 are isometric). Then we have the conditions of the extension theorem, so there exists an isometry $F : X_1 \to X_2$ such that $F|_{V_1} = f$ and $F(U_1) = U_2$, that is, U_1 is isometric to U_2. Now assume the cancellation theorem and suppose we have X_1, X_2, U_1, U_2, V_1, V_2 as in the extension theorem. Then since V_1 is isometric to V_2, it follows that $U_1 \oplus V_1$ is isometric to $U_2 \oplus V_2$, so by the cancellation theorem, U_1 is isometric to U_2. So there must be an isometry between U_1, U_2. Let this isometry be f_U. Then $F = f_U + f$, where $f : V_1 \to V_2$ comes from the assumptions of the extension theorem, maps $X_1 = U_1 \oplus V_1$ to $X_2 = U_2 \oplus V_2$, is an isometry, $F|_{V_1} = f$, and $F(U_1) = f_U(U_1) = U_2$, so the extension theorem holds.

We are now ready for Witt's Theorem (we follow the statement in [26], Theorem 7.4).

Theorem 4.3.3 (Witt's Theorem)**.** *Suppose that U is a subspace of V and that the map $f : U \to U$ is an isometry. Then there is an isometry $g : V \to V$ such that $g(u) = f(u)$ for all $u \in U$ if and only if $f(U \cap \operatorname{rad}(V)) = f(U) \cap \operatorname{rad}(V)$.*

Sketch of proof, following [26]. Here we provide an outline of D.E. Taylor's proof. Note that some assertions are given as lemmata in [26].

If we have an isometry $g : V \to V$ such that $g(u) = f(u)$ for all $u \in U$, then it follows that $f(U \cap \operatorname{rad}(V)) = f(U) \cap \operatorname{rad}(V)$. Now, suppose that $f(U \cap \operatorname{rad}(V)) = f(U) \cap \operatorname{rad}(V)$. The proof in the other direction is more involved and works through several cases. In each case, we construct the desired isometry g.

Firstly, assume that $\operatorname{rad}(V) \nsubseteq U$ and $\operatorname{rad}(V) \nsubseteq f(U)$. Choose a subspace W to be the complement to both $U \cap \operatorname{rad}(V)$ and $f(U) \cap \operatorname{rad}(V)$ in $\operatorname{rad}(V)$. Then $U + \operatorname{rad}(V) = U \oplus W$ and $f(U) + \operatorname{rad}(V) = f(U) \oplus W$, so $f + \mathbf{1}_W$ is an isometry extending f (here $\mathbf{1}_W$ indicates the identity function on W).

Now assume that $\operatorname{rad}(V) \subseteq U$ and $\operatorname{rad}(V) \subseteq f(U)$. If $U = \operatorname{rad}(V)$ and if W is the complement to U in V, then $f + \mathbf{1}_W$ is an isometry extending f. Hence, we assume from now on that $U \neq \operatorname{rad}(V)$. In this case, consider a hyperplane H of U containing $\operatorname{rad}(V)$ and let f' be the restriction of f to H. Then f' has an extension $g' : V \to V$. By replacing f by $f'^{-1} f$, we can assume that f fixes H.

If f fixes U, then $g = \mathbf{1}$ and we are done. So now suppose f does not fix U. Then we construct $P = (f - \mathbf{1})(U)$, a one-dimensional subspace of V. We can show that $H \subseteq P^\perp$, so then $U \subseteq P^\perp$ if and only if $f(U) \subseteq P^\perp$. If $U \nsubseteq P^\perp$, then $U \cap P^\perp = f(U) \cap P^\perp = H$. Let W be the complement to H in $P^\perp$. Then $V = W \oplus U$, so $\mathbf{1}_W + f$ is an isometry of V extending f.

Now assume $U \subseteq P^\perp$ and $f(U) \subseteq P^\perp$, then $P \subseteq P^\perp$. If $U \neq f(U)$, then construct X, the complement to both U and $f(U)$ in $U + f(U)$. Let W be the complement to $U + f(U)$ in $P^\perp$ and let $S = W + X$. Then $P^\perp = S \oplus U = S \oplus f(U)$, so $\mathbf{1}_S + f$ is an isometry of $P^\perp$ extending f. If $U = f(U)$, then let S be the complement to U in $P^\perp$ and we see that $\mathbf{1}_S + f$ is an isometry of $P^\perp$. In the two previous cases, $\mathbf{1}_S + f$ is the identity on a hyperplane of $P^\perp$ containing $\operatorname{rad}(V)$.

Now suppose $U = P^\perp = f(U) \neq V$. Then $P = \langle u \rangle = \langle f(v) - v \rangle$ for some $v \in U$, so $P \subseteq P^\perp$ and u is self-orthogonal. Let L be a two-dimensional subspace such that $P \subseteq L$ but $L \nsubseteq P^\perp$. Then L is a non-degenerate subspace, so $L = \langle u, w \rangle$, for (u, w) a hyperbolic pair. We see that $w \notin \operatorname{rad}(V)$, so $\langle w \rangle^\perp$ is a hyperplane of V and $L^\perp$

is a hyperplane of U, hence $\langle w\rangle^{\perp} \cap U = L^{\perp}$. So define $Y = f(L^{\perp})$, then $\langle w\rangle + Y$ is a hyperplane of V containing $\mathrm{rad}(V)$ but not $f(U)$, so $\langle w\rangle + Y = \langle w'\rangle^{\perp}$ for some $w' \notin U$. Hence $\langle f(u), w'\rangle$ is a non-degenerate subspace and $Y = \langle f(u), w'\rangle^{\perp}$. Then there exists a self-orthogonal vector w'' such that $\langle f(u), w'\rangle = \langle f(u), w''\rangle$, where $(f(u), w'')$ is a hyperbolic pair. Define a map $g : \langle w\rangle \to V$ by $aw \mapsto aw''$. Then g is an isometry, $U = \langle u\rangle \oplus L^{\perp}$, $V = \langle w\rangle \oplus U$, so $g + f$ is an isometry of V. Q.E.D.

Orbits of the hyperbolic quadric on planes

In order to determine the orbits of the hyperbolic quadric on planes, we must first identify the *isometry group* of the hyperbolic quadric. The isometry group will be the set of all isometries of the quadric. With respect to the standard basis in $\mathrm{PG}(3,\mathbb{R})$, let $M = \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$ be the matrix associated with the hyperbolic quadric $\mathcal{Q}$. Take $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ to be an arbitrary matrix in $\mathrm{PGL}(4,\mathbb{R})$. If g is in the isometry group of $\mathcal{Q}$, then $gMg^{\top} = M$. To find the conditions on such a matrix g, we compute

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & O \\ O & -I \end{bmatrix} \begin{bmatrix} A^{\top} & C^{\top} \\ B^{\top} & D^{\top} \end{bmatrix} = \begin{bmatrix} (AA^{\top} - BB^{\top}) & (AC^{\top} - BD^{\top}) \\ (CA^{\top} - DB^{\top}) & (CC^{\top} - DD^{\top}) \end{bmatrix}.$$

Thus we have the following proposition on the properties of elements in the isometry group of $\mathcal{Q}$:

Proposition 4.3.4. *A matrix* $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is in the isometry group of* $\mathcal{Q}$ *if it is invertible and if the following four conditions hold:*

1. $(AA^{\top} - BB^{\top}) = I$,
2. $(AC^{\top} - BD^{\top}) = 0$, *and*
3. $(CC^{\top} - DD^{\top}) = -I$.

If a matrix in $\mathrm{PGL}(4,\mathbb{R})$ satisfies these conditions, then the matrix is in the isometry group of $\mathcal{Q}$.

Now we will have a little diversion through some necessary background for the reasoning which follows. Let U be a subspace of a vector space V. The *perp* or *orthogonal complement* of U is the subspace containing all the elements of V that are orthogonal to all the elements of U with respect to a form B. That is, $U^{\perp} = \{v \in V : B(u, v) = 0 \text{ for every } u \in U\}$.

Example 4.3.5. *For an example relevant to our case, let* $P \in \mathrm{PG}(3,\mathbb{R})$ *be a point. Then* $P^{\perp}$ *will be the set of points* $Q \in \mathrm{PG}(3,\mathbb{R})$ *such that* $B(P,Q) = 0$, *hence* $P^{\perp} = \{Q \in \mathrm{PG}(3,\mathbb{R}) : B(P,Q) = 0\}$. *Relating this to the associated matrix* M, *these are the points* $Q \in \mathrm{PG}(3,\mathbb{R})$ *such that* $PMQ^{\top} = 0$. *This set defines a plane, so we see that the perp of a point is a plane.*

Consider the point $P = (1, 0, 0, 0)$ *and the bilinear form associated with the hyperbolic quadric. Then points* $Q = (w, x, y, z)$ *such that* $B(P,Q) = 0$ *satisfy*

$PMQ^\top = 0$, that is

$$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}w\end{bmatrix} = 0.$$

Thus, it follows that $P^\perp = \{Q \in \mathrm{PG}(3,\mathbb{R}) : Q = (0,x,y,z)\}$ and this is the plane $\{w = 0\}$.

If π is a plane, then $\pi^\perp = \{Q \in \mathrm{PG}(3,\mathbb{R}) : B(P,Q) = 0$ for every $P \in \pi\}$. Hence $\pi^\perp$ is a point (we will see this later in Claim 8).

Let U be a subspace of $\mathrm{PG}(3,\mathbb{R})$. U is a *non-degenerate* subspace if $U \cap U^\perp = \varnothing$. Clearly, if U is non-degenerate, then $U^\perp$ is non-degenerate, and vice versa.

Consider $\mathcal{Q}$. Either a point is on $\mathcal{Q}$ or it is not on $\mathcal{Q}$. This means that if P is a point on $\mathcal{Q}$, then $P \subseteq P^\perp$, but if P is a point not on $\mathcal{Q}$, then $P \cap P^\perp = \varnothing$. Hence the perp of the non-degenerate planes are the points not on $\mathcal{Q}$. Similarly, the perp of a degenerate plane is a point on $\mathcal{Q}$. This means the degenerate planes are in a one-to-one correspondence with the points on $\mathcal{Q}$ because the points on $\mathcal{Q}$ always meet in the intersection of two lines of the two reguli generating $\mathcal{Q}$.

Claim 7. *The isometry group of $\mathcal{Q}$ acts transitively on the set of points of $\mathcal{Q}$.*

Proof. Consider $v_0 = (1,0,1,0)$. Clearly, $1^2 + 0^2 - 1^2 + 0^2 = 0$, so $v_0 \in \mathcal{Q}$. Let $v' = (v'_1, v'_2, v'_3, v'_4)$ be any point in $\mathcal{Q}$, then $(v'_1)^2 + (v'_2)^2 - (v'_3)^2 - (v'_4)^2 = 0$, so $(v'_1)^2 + (v'_2)^2 = (v'_3)^2 + (v'_4)^2$. Let $\lambda = ((v'_1)^2 + (v'_2)^2)^{-\frac{1}{2}}$ and note that λ always exists since $(v'_1)^2 + (v'_2)^2$ is positive. Then without loss of generality (since in projective space we have scalar equivalence of points), let $v = (v_1, v_2, v_3, v_4) = \lambda v'$, so that $v_1^2 + v_2^2 = v_3^2 + v_4^2 = 1$.

Now consider $g = \begin{bmatrix} v_1 & v_2 & 0 & 0\\ v_2 & -v_1 & 0 & 0\\ 0 & 0 & v_3 & v_4\\ 0 & 0 & v_4 & -v_3\end{bmatrix}$. We see that

$$\begin{bmatrix} v_1 & v_2\\ v_2 & -v_1\end{bmatrix}\begin{bmatrix} v_1 & v_2\\ v_2 & -v_1\end{bmatrix} = \begin{bmatrix} v_1^2 + v_2^2 & v_1v_2 - v_1v_2\\ v_1v_2 - v_1v_2 & v_2^2 + v_1^2\end{bmatrix} = I$$

and similarly for $\begin{bmatrix} v_3 & v_4\\ v_4 & -v_3\end{bmatrix}$. Hence, g is in the isometry group of $\mathcal{Q}$ because g satisfies Proposition 4.3.4.

Furthermore,

$$v_0^g = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix} v_1 & v_2 & 0 & 0\\ v_2 & -v_1 & 0 & 0\\ 0 & 0 & v_3 & v_4\\ 0 & 0 & v_4 & -v_3\end{bmatrix} = \begin{bmatrix}v_1 & v_2 & v_3 & v_4\end{bmatrix} = v,$$

hence we can map between any two points in $\mathcal{Q}$ by an element of the isometry group, so the isometry group of $\mathcal{Q}$ acts transitively on $\mathcal{Q}$. Q.E.D.

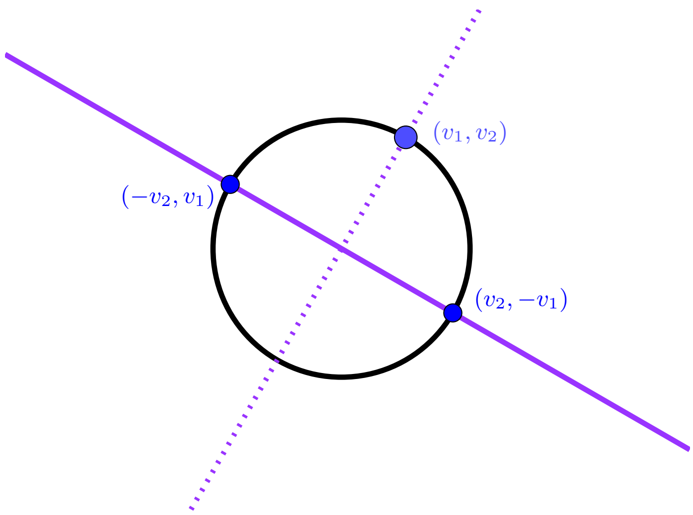

Figure 4.2: Finding the perpendicular points to (v_1, v_2).

Remark 4.3.6. *It may not be immediately obvious to the reader how the matrix g was chosen. Starting with any matrix $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the simplest step was to set $B = C = 0$. By choosing the first row of A to be the vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$ and the first row of D to be the vector $\begin{bmatrix} v_3 & v_4 \end{bmatrix}$, we ensured that $\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} g = v$. Then we only needed to ensure that $AA^\top = DD^\top = I$. For the matrix A, this would mean finding a vector $\begin{bmatrix} w_1 & w_2 \end{bmatrix}$ such that $w_1^2 + w_2^2 = 1$ and $(v_1, v_2) \cdot (w_1, w_2) = 0$, where $\cdot$ indicates the standard Euclidean dot product. Recognising that the points (v_1, v_2) and (w_1, w_2) are both on the unit circle, it is clear geometrically that (w_1, w_2) is the intersection of the perpendicular to the line $y = \frac{v_2}{v_1}x$ with the unit circle. The perpendicular to this line is $y = -\frac{v_1}{v_2}$, which has two intersections with the unit circle, $(v_2, -v_1)$ and $(-v_2, v_1)$ (see Figure 4.2). Similarly for the matrix D.*

Remark 4.3.7. *Alternatively, we can prove Claim 7 using Witt's Theorem and Lemma 4.3.8 below. Once we have the isometry f as in Lemma 4.3.8, Claim 7 follows from Witt's Theorem because f extends as an isometry to the whole space.*

Lemma 4.3.8 (See [12]). *Let U_1 and U_2 be* totally isotropic subspaces *of a vector space V with a bilinear form B, that is, for any u, $v \in U_1$, $B(u, v) = 0$ and similarly for U_2. Then any bijective linear map f between them is an isometry.*

Proof. Let u, $v \in U_1$ and let $f : U_1 \to U_2$ be a bijective linear map. Then

$$B(f(u), f(v)) = 0 = B(u, v).$$

Q.E.D.

As a result of Claim 7, we have the following corollary:

Corollary 4.3.9. *There is only one orbit on degenerate planes (the perps of the points on $\mathcal{Q}$).*

Claim 8. *There are two orbits on non-degenerate planes (the perps of the points not on $\mathcal{Q}$).*

Proof. We can use Sylvester's Law (Theorem 4.1.2) to show that there are at least two orbits of the stabiliser of the hyperbolic quadric on non-degenerate planes, but here we provide a direct proof. Let π_1 be the plane $z = 0$ and let $\{e_1, e_2, e_3\}$ be a basis for π_1, where e_i is the vector with 1 in the ith coordinate and 0 elsewhere. Then $\mathcal{Q}$ restricted to the subspace π_1 has associated matrix $M_{\pi_1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$. Let π_2 be the plane $x = 0$ and let $\{e_1, e_3, e_4\}$ be a basis for π_2. Then $\mathcal{Q}$ restricted to the subspace π_2 has associated matrix $M_{\pi_2} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$.

Both π_1 and π_2 are non-degenerate planes and we can verify this by checking that the orthogonal complements of these planes are not points in $\mathcal{Q}$. Firstly, $\pi_1^\perp$ will be some point $P_1 \in \mathrm{PG}(3, \mathbb{R})$ such that

$$P_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \begin{pmatrix} w \\ x \\ y \\ 0 \end{pmatrix} = 0,$$

since every point in π_1 will have w, x, y free and $z = 0$. Then $P_1 = (0, 0, 0, 1)$ and $0^2 + 0^2 - 0^2 - 1^2 = -1 \neq 0$, so $P_1 \notin \mathcal{Q}$. Hence π_1 is a non-degenerate plane. Similarly, $\pi_2^\perp$ will be some point $P_2 \in \mathrm{PG}(3, \mathbb{R})$ such that

$$P_2 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \begin{pmatrix} w \\ 0 \\ y \\ z \end{pmatrix} = 0,$$

since every point in π_2 will have w, y, z free and $x = 0$. We see then that $P_2 = (0, 1, 0, 0)$ and $0^2 + 1^2 - 0^2 - 0^2 = 1 \neq 0$, so $P_2 \notin \mathcal{Q}$. Hence π_2 is likewise a non-degenerate plane.

We want to know whether there exists a matrix g in the isometry group of $\mathcal{Q}$ such that π_1 acted on by g is sent to π_2, that is, whether we can map π_1 to π_2 by an element of the isometry group of $\mathcal{Q}$. It follows by definition that an isometry exists between two subspaces if and only if an isometry exists between their perps. Hence, we consider whether it is possible to map $\pi_1^\perp = P_1$ to $\pi_2^\perp = P_2$ by an element in the isometry group.

We know how elements in the isometry group look, they satisfy Proposition 4.3.4. Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be in the isometry group of $\mathcal{Q}$. If g maps P_1 to P_2, then

$$\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ gives us the last row of $\begin{bmatrix} C & D \end{bmatrix}$, we want to know if the last row of $\begin{bmatrix} C & D \end{bmatrix}$ can be $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$. Suppose this is true. Because we are working in $\mathrm{PGL}(4, \mathbb{R})$, we consider whether the last row of $\begin{bmatrix} C & D \end{bmatrix}$ can be $\begin{pmatrix} 0 & \lambda & 0 & 0 \end{pmatrix}$. Consider

$$g = \begin{bmatrix} & & & \\ & & & \\ c_1 & c_2 & d_1 & d_2 \\ 0 & \lambda & 0 & 0 \end{bmatrix}.$$

Then $CC^\top = \begin{bmatrix} (c_1^2 + c_2^2) & \lambda c_2 \\ \lambda c_2 & \lambda^2 \end{bmatrix}$ and $DD^\top = \begin{bmatrix} (d_1^2 + d_2^2) & 0 \\ 0 & 0 \end{bmatrix}$. Hence $CC^\top - DD^\top = \begin{bmatrix} (c_1^2 + c_2^2) - (d_1^2 + d_2^2) & \lambda c_2 \\ \lambda c_2 & \lambda^2 \end{bmatrix}$. Then in order to satisfy the conditions of Proposition 4.3.4, we have $CC^\top - DD^\top = -I$ if $c_2 = 0$ (we can do this and maintain invertibility) and $\lambda^2 = c_1^2 - (d_1^2 + d_2^2)$ is such that $\lambda^2 < 0$. This is only possible over the complex numbers. So over the real numbers, there are at least two orbits on non-degenerate planes.

We have now shown that for two chosen non-degenerate planes π_1 and π_2 in $\mathrm{PG}(3, \mathbb{R})$, there does not exist an isometry between them. This means that there are at least two orbits on non-degenerate planes. By Sylvester's Law (Theorem 4.1.2), there are only two non-empty quadrics over $\mathbb{R}^3$, having the signatures $\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$, respectively. This tells us that over $\mathbb{R}^3$ there are only two quadrics, up to isometry. But this does not mean that there are only two quadrics up to isometry when we view these quadrics as conics embedded in $\mathbb{R}^4$.

To show this, we will need to appeal to Witt's Extension Theorem (Theorem 4.3.2). Without loss of generality, suppose π is a plane that is isometric to π_1 when considered as quadratic spaces over $\mathbb{R}^3$. Let X_1, X_2 be $\mathbb{R}^4$ equipped with a form such that $X_1 = \pi_1 \oplus V_1$, for some V_1, and $X_2 = \pi \oplus V_2$, for some V_2. Since π_1 is isometric to π and X_1 is isometric to X_2 (because, in fact, $X_1 = X_2$), it follows from Witt's Theorem that π_1 and π are isometric as quadrics in $\mathbb{R}^3$. By Theorem 4.3.2, this isometry extends, so there is an isometry of X_1 mapping π_1 to π. Hence, there are exactly two orbits of the stabiliser of the hyperbolic quadric on planes. Q.E.D.

Conditions on lines

In the previous section, we considered the requisite conditions on the fixed plane in order to determine the conic generated by skew projection – that is, a plane of a certain type intersects the quadric in a certain conic. We now, alternatively, consider the conditions on skew lines. We have three mutually skew lines and a plane. We want to know how the lines are configured with respect to the plane – how many lie in the plane, how many intersect the plane in one point, and how many do not intersect the plane at all.

In Section 4.3, we showed that there is only one orbit on degenerate planes but we did not give a representative for the degenerate planes. Take the point $v_0 = (1, 0, 1, 0)$, which is a point in the quadric since $1^2 + 0^2 - 1^2 - 0^2 = 0$. The perp of this point is the plane $v_0^\perp = \{(w, x, y, z) \in \mathrm{PG}(3, \mathbb{R}) : w = y\}$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ w \\ z \end{bmatrix} = 0,$$

where (w, x, w, z) is a point in $v_0^\perp$. Then $v_0^\perp$ intersects the quadric in the set of points

$$\begin{aligned} &\{(w, x, w, z) \in v_0^\perp : w^2 + x^2 - w^2 - z^2 = 0\} \\ =&\{(w, x, w, z) \in v_0^\perp : x^2 - z^2 = 0\} \\ =&\{(w, x, w, z) \in v_0^\perp : x = \pm z\} \\ =&\{(w, x, w, z) \in v_0^\perp : x = z\} \cup \{(w, x, w, z) \in v_0^\perp : x = -z\} \\ =&\{(w, x, w, x)\} \cup \{(w, x, w, -x)\}, \end{aligned}$$

which defines two lines with intersection $(1, 0, 1, 0)$. Hence, the degenerate planes intersect the quadric in two intersecting lines (a degenerate conic).

For the non-degenerate planes, π_1 intersects the quadric in the set of points $w^2 - y^2 - z^2 = 0$, which is the circle in the yz-plane with radius w, and π_2 intersects the quadric in the set of points $w^2 + x^2 - y^2 = 0$, which is the circle in the wx-plane with radius y. Hence, the non-degenerate planes intersect the quadric in a non-degenerate conic.

Now, as demonstrated in Section 3.1, a line either lies in a plane or intersects it in one point. So we know that none of the three lines can be disjoint from the fixed plane. By definition of their mutual skewness, no two of the lines can lie in the same plane. This leaves us with two cases: 1) one line lies in the fixed plane and the other two intersect it in one point and 2) all three lines intersect the fixed plane in one point. One further fact about conics worth mentioning here is that non-degenerate conics are determined by five points, no three collinear ([6], page 87). Hence, non-degenerate conics do not contain three collinear points and by extension, they do not contain lines.

In the first case, one of the skew lines is intersecting the plane in a line, so the intersection of the quadric with the plane cannot be a non-degenerate conic because non-degenerate conics do not contain lines. In this case, skew projection generates a degenerate conic. Furthermore, the type of degenerate conic is two intersecting lines because if one set of transversals is intersecting the plane in a line, the other set of transversals must be intersecting in a line which intersects the one in the plane.

In the second case, there is one more subtlety to consider. If the three skew lines intersect the plane in three collinear points, then the quadric is intersecting the plane in at least three collinear points, so the quadric must be intersecting a degenerate plane because a non-degenerate conic (generated by the intersection of a quadric with a non-degenerate plane) does not contain three collinear points. Moreover, since the three skew lines meet the plane in collinear points, their transversal must lie in the plane. Then any two additional lines in their regulus must also meet the plane, so the quadric is meeting the plane in two intersecting lines. Hence in the case that the three lines intersect the plane in three collinear points, we again generate a degenerate conic (two intersecting lines).

In the case that the three points of intersection are non-collinear, let l, m, n be the three skew lines and let L, M, N be the respective points of intersection of each line with a plane π. Suppose, in order to gain a contradiction, that π is degenerate. Then we know that π intersects the quadric in two lines, so two of L, M, N must be contained in a line of the quadric in π. Without loss of generality, suppose L, M are contained in this line. Since LM is a line, $(LM)^\perp = LM$ is also a line. The following subspace identity holds for projective subspaces: if A, B are subspaces, then $\langle A, B\rangle^\perp = A^\perp \cap B^\perp$. Using this, we see that $L = \langle l, \pi\rangle$, so $L^\perp = \langle l, \pi\rangle^\perp = l^\perp \cap \pi^\perp$. Similarly, $M = \langle m, \pi\rangle$, so $M^\perp = \langle m, \pi\rangle^\perp = m^\perp \cap \pi^\perp$. Then $LM = \langle L, M\rangle = \langle L, M\rangle^\perp = L^\perp \cap M^\perp = (l^\perp \cap \pi^\perp) \cap (m^\perp \cap \pi^\perp)$, so $\pi^\perp$ is a point on LM. Moreover, since $LM = L^\perp \cap M^\perp$, we have that LM is a line in the plane $L^\perp$. Also, $L = l \cap \pi$, so $L^\perp = (l \cap \pi)^\perp = \langle l, \pi^\perp\rangle$, hence l is a line in the plane $L^\perp$. Thus $L^\perp = \langle LM, l\rangle$. Similarly, $M^\perp = \langle LM, l\rangle$. Hence $L^\perp = M^\perp$, so the lines l, m lie in the same plane, which is a contradiction, since l, m are skew. We conclude, then, that when three skew lines intersect a plane in three non-collinear points, a non-degenerate conic is generated. Every conic is an intersection of the hyperbolic quadric with a plane (Proposition 3.3.1). Take a non-degenerate conic, choose three non-collinear points on the conic (which necessarily exist), choose three lines from the same regulus through these points and these are the three skew lines required to generate this conic. Hence we can obtain every non-degenerate conic when the three skew lines intersect the plane in three non-collinear points.

This gives us the following theorem:

Theorem 4.3.10. *Given a skew projection in* $\mathrm{PG}(3, \mathbb{R})$ *(as outlined in Section 1.3), if the three skew lines meet the plane in three non-collinear points, then skew projection generates a non-degenerate conic. Otherwise, skew projection generates a degenerate conic in the form of two intersecting lines.*

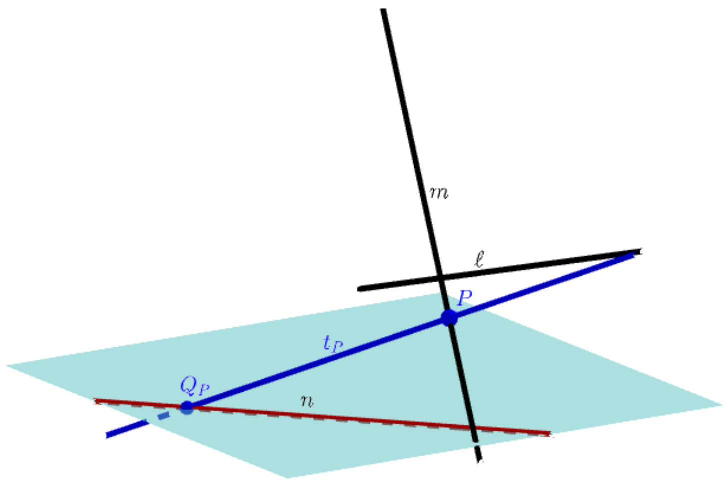

Figure 4.3: Theorem 4.3.10 (first degenerate case, one skew line lies in the plane).

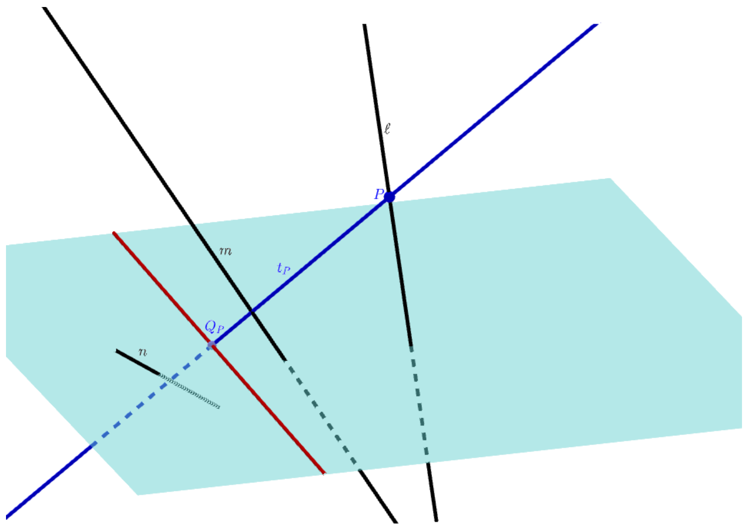

Figure 4.4: Theorem 4.3.10 (second degenerate case, the three intersecting points of the skew lines are collinear).

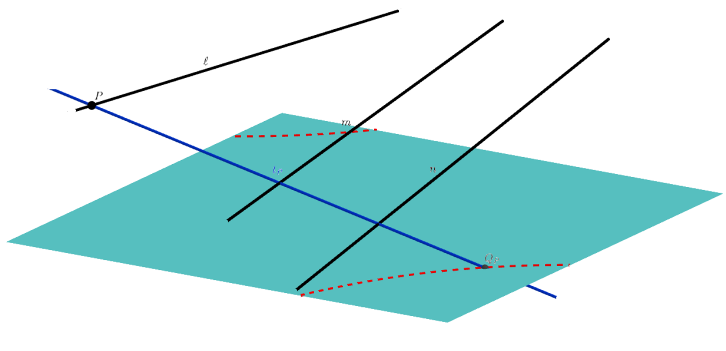

Figure 4.5: Theorem 4.3.10 (non-degenerate case, the three intersecting points of the skew lines are non-collinear).

Chapter 5

Concluding Remarks

5.1 Summary

In this thesis, we investigated skew projection and in particular, we were interested in determining how skew projection generates a conic. After working through the relevant background material, this aim was achieved in Chapter 3. Here we proved the existence of skew projection (Section 3.1) and, by proving that the action of $\mathrm{PGL}(3, \mathbb{R})$ on triples of skew lines is transitive (Proposition 3.2.1), we proved the existence and uniqueness of a hyperbolic quadric generated by three skew lines and their regulus (Theorem 3.2.4).

After proving Sylvester's Law of Inertia (Theorem 4.1.2), we ascertained that there are two projective non-degenerate quadrics, up to projective equivalence, the hyperbolic quadric and the elliptic quadric (Theorem 4.1.1). The hyperbolic quadric is the only projective non-degenerate quadric containing lines, hence we concluded that three skew lines and their transversals generate a hyperbolic quadric. Thus, the conic generated by skew projection is the intersection of a plane with a hyperbolic quadric.

The main result of this thesis was the determination of the types of conics generated by skew projection. To do this, we determined the orbits of the stabiliser of the hyperbolic quadric on degenerate and non-degenerate planes. We discussed Witt's Theorem (Theorem 4.3.3) which, along with Sylvester's Law, enabled us to ascertain that there are three orbits of the hyperbolic quadric on planes: two orbits on non-degenerate planes and one orbit on degenerate planes. We then considered, alternatively to the conditions on planes, the conditions on three skew lines and ascertained that a degenerate conic is generated when three skew lines meet an underlying plane in three collinear points or when one of the three skew lines lies in the plane and that a non-degenerate conic is generated when three skew lines meet an underlying plane in three non-collinear points.

5.2 Further considerations

Further to this thesis, there is still much to be discovered regarding skew projection. The proof of the set up of skew projection in Section 3.1 relied solely on dimension arguments which hold in all projective spaces. It follows, then, that we can consider skew projection over spaces other than the real numbers. There are three cases that we consider in these concluding remarks.

Skew projection over finite fields

The 16 Point Theorem of Dandelin-Gallucci (Theorem 2.4.1) gives the provisor that it is necessary for skew lines and their transversals to be taken over a field in order to generate a unique regulus, so it follows that a unique regulus can be generated over finite fields (and the complex numbers and other fields, which we will discuss later). With a unique regulus, the remaining arguments for generating a unique hyperbolic quadric will hold over finite fields, except possibly over fields of characteristic two.

The main issue in the case of finite fields is, of course, fields of characteristic two. This could certainly be a basket case, especially since quadratic forms are determined by the number of positive and negative ones in diagonalised form (Sylvester's Law 4.1.2), which of course would not hold in characteristic two, where positive numbers equal negative numbers. A unique regulus will be generated, the issue is whether it will determine a quadratic form. In fact, in characteristic two the bilinear form will be a *symplectic* form, one which is both symmetric and alternating simultaneously.

There is great potential here – we need not limit ourselves to the finite case, even. There are many more general fields which could be considered. The 2-adics, for instance, are an infinite field with characteristic two which could work a lot more 'nicely' than finite fields of characteristic two.

Skew projection over the complex numbers

Most obviously, this thesis considered skew projection over the real numbers, yet it is also possible to consider skew projection over the complex numbers $\mathbb{C}$. As aforementioned, as a result of the Dandelin-Gallucci Theorem, we know that a unique regulus will be generated over the complex numbers. The remaining arguments easily extend to the complex numbers and we will once again generate a unique hyperbolic quadric.

Whilst Sylvester's Law of Inertia (Theorem 4.1.2) will still hold, when we classify the non-empty non-degenerate quadrics of $\mathrm{PG}(3, \mathbb{C})$ (Theorem 4.1.1), the first case $w^2 + x^2 + y^2 + z^2 = 0$ will be no longer empty. Even as far as it concerns conics, the empty conic $x^2 + y^2 = -1$ will be no longer empty when solutions over $\mathbb{C}$ are considered.

The alternative formulation of Sylvester's Law of Inertia (Theorem 4.1.6) has a natural extension to $\mathbb{C}$, as does our classification of affine quadrics in Section 4.2.1. For further information, the reader is directed to the treatment of the classification of affine quadrics over the complex numbers in [4] (Proposition 15.3.1).

An interesting consequence of skew projection over $\mathbb{C}$ will be found when determining the orbits of the quadric on planes. Firstly, recall that in Claim 8, we argued that there are two orbits of the quadric on non-degenerate planes. However, our argument would not hold over the complex numbers, where the desired isometry g between the two non-degenerate planes will exist.

Skew projection over Hermitian spaces

Another interesting path would be finding analogous constructions in Hermitian spaces, that is, vector spaces endowed with a Hermitian form as opposed to a quadratic form. We can consider Hermitian spaces over $\mathbb{C}$ or $\mathbb{F}_{q^2}$ (fields of order q^2). Whereas the points x in a quadratic form are such that $xMx^\top = 0$, for M the associated matrix of some quadratic form $\mathcal{Q}$, the points which comprise a Hermitian form are such that $xM\overline{x}^\top = 0$, for M the associated matrix of some Hermitian form $\mathcal{H}$, where $\overline{x}$ indicates the complex conjugate of the point x. A Hermitian matrix M has real eigenvalues and is equal to its conjugate transpose, that is, $M = \overline{M}^\top$.

Consider the following set-up over the complex numbers. Let $l = \begin{bmatrix} I & O \end{bmatrix}$, $m = \begin{bmatrix} O & I \end{bmatrix}$, and $n = \begin{bmatrix} I & I \end{bmatrix}$ be three skew lines, where we allow the lines to take complex values. Let P be a point on l, where possibly P could be complex-valued. Then $P = (1, v, 0, 0)$ for some $v \in \mathbb{C}$.

We want an equation for the unique transversal t_P to the three skew lines through P (note once again that t_P is still unique since the proof in Section 3.1 of the uniqueness of a transversal to three skew lines through a given point relied solely on dimension arguments which hold in any projective space). Firstly, let's compute the span of P and m:

$$\langle P, m\rangle = \begin{bmatrix} 1 & v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix},$$

where (a, b, c, d) is any point in $\mathrm{PG}(3, \mathbb{C})$. This yields the plane π with equation $vw = x$. To find the intersection of π with the line n, we must solve the three equations $vw = x$, $w = y$, and $x = z$ simultaneously. Then the intersection $\pi \cap n = N_P = (1, v, 1, v)$. The transversal t_P is the span of P and N_P, so $t_P = \langle (1, v, 1, v), (1, v, 0, 0)\rangle$. Hence any point on t_P has the form $(1, v, b, bv)$ for some $b \in \mathbb{C} \cup \{\infty\}$.

Now let $\mathcal{U} = \begin{bmatrix} O & B \\ -\overline{B}^\top & O \end{bmatrix}$ be a Hermitian matrix. Firstly, we consider P to be a point on the real part of the line l, so $v \in \mathbb{R}$. We want to show that the transversal $t_P \subseteq \mathcal{U}$.

There is still much to discover about skew projection, but at least this thesis, in teasing out the situation over the real numbers, can in some small way lay the foundation for further studies of this seemingly magical and all together beautiful geometric construction.

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